

# Dynamic Information Aggregation: Learning from the Past\*

Zhen Huo  
Yale University

Marcelo Pedroni  
University of Amsterdam

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## Abstract

With dispersed information, how much can agents learn from past endogenous aggregates such as prices or output? In a rational-expectations equilibrium, if general equilibrium effects are strong enough, aggregates no longer perfectly reveal underlying fundamentals. In this confounding regime, the effects of informational frictions are persistent over time, and the aggregate outcome displays an initial underreaction followed by a delayed overreaction relative to its perfect-information counterpart. In a standard New Keynesian model, we show that the confounding regime is more likely to arise under a dovish monetary policy rule.

Keywords: Information Aggregation, Dispersed Information, Monetary Policy

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\*Huo: Yale University, 28 Hillhouse Ave, New Haven, CT, 06510, US, zhen.huo@yale.edu. Pedroni: University of Amsterdam, PO Box 15867, 1001 NJ Amsterdam, Netherlands, m.pedroni@uva.nl. Acknowledgements: We are thankful to Yan Xiong for discussing our paper in the 2019 PHBS Workshop in Macroeconomics and Finance. The paper benefits from comments from Marios Angeletos, Ryan Chahrour, Simas Kucinskas, Keith Kuester, Jeniffer La'O, Giacomo Rondina, Christian Stoltenberg, Todd Walker, and Pengfei Wang. We would also like to thank the seminar participants at PHBS, University of Bonn, TI Macro Conference in Rotterdam. We thank Flint O'Neil for providing excellent research assistance. Lastly, we thank Zhao Han, Fei Tan, and Jieran Wu for generously sharing their computational results with us.

# 1. INTRODUCTION

In this paper, we revisit a number of classical questions in the information literature: how much can agents learn from endogenous aggregate outcomes about underlying fundamentals? How does the informativeness of aggregate statistics depend on the degree strategic complementarity? Can monetary policy generate persistent real effects when the history of inflation is perfectly observed? We explore these questions in economies with persistent information and strategic interactions between agents.

We start with a canonical beauty-contest model in which agents care about some persistent economic fundamental and also an endogenous aggregate outcome. We assume agents receive two types of signals: private noisy signals about current fundamentals and the perfect observation of the history of past aggregate outcomes. The lack of observation of the current aggregate outcome captures the idea that firms and households often make their production and consumption decisions before contemporaneous aggregate statistics are available. We show that in a rational expectations equilibrium, two possible regimes can arise endogenously: a revealing regime and a confounding regime.<sup>1</sup> In the revealing regime, the history of outcomes perfectly aggregate information, and agents can infer the underlying fundamental without error. This is in line with the conventional wisdom that, even with dispersed information, prices or output help eliminate uncertainty about the aggregate fundamental. However, there also exists a confounding regime in which agents make persistent forecast errors, and consensus about the fundamental is never achieved.

Whether the confounding regime arises depends on the degree of strategic complementarity between agents which captures the strength of GE feedback effects, and on the precision of private signals about the fundamental. Endogenous outcomes aggregate information only if agents respond to their private signals which are direct sources of information. The more agents rely on past aggregate outcomes in their inference, the less information contained in private signals can be encoded into individuals' actions, resulting in potential information dilution. The extent to which agents rely on the history of aggregate outcomes is decreasing in the precision of private signals and increasing in the need to coordinate with other agents.<sup>2</sup> In an environment with persistent information, a sufficient amount of information dilution can make the process of the aggregate outcome non-invertible. This insight extends to various environments with multiple signals, forward-looking beauty-contest games, and multivariate systems.

The dynamics of aggregate outcomes also differ in a significant way under these two regimes. In the revealing regime, in response to an innovation to the fundamental, the aggregate outcome underreacts on impact, since information about the current fundamental is dispersed. Afterwards,

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<sup>1</sup>The notion of confounding dynamics that results from a non-invertible process was first introduced in an early version of [Rondina and Walker \(2020\)](#).

<sup>2</sup>See a more detailed discussion on this point in [Morris and Shin \(2002\)](#), [Angeletos and Pavan \(2007\)](#), and [Huo and Pedroni \(2020\)](#).

the aggregate outcome exactly follows the fundamental since, by observing past aggregate outcomes, agents reach common knowledge about the innovation. In contrast, in the confounding regime, past outcomes are not sufficient for agents to infer the underlying innovations, and agents' forecast errors are persistent over time. Still, the observation of past aggregate outcomes does not allow for the systematic under- or overestimation of the fundamental. As a result, the aggregate outcome fluctuates around the fundamental, alternating between overreaction and underreaction. Furthermore, we prove that the process for the aggregate outcome is complex in the sense that it does not permit a finite-state representation.

An implication of the two informational regimes is that there exists a “kink” on how the informativeness of aggregate outcomes is related to the structural parameters. Within the revealing regime, a local variation of parameters only affects the impact response of the aggregate outcome but does not change its response afterwards nor its invertibility. In response to a large enough shift in the parameters, the economy enters the confounding regime, in which case either a larger strategic complementarity or a lower signal precision decrease the informativeness of the aggregate outcome.

We explore two applications of our results. We first revisit the classical monetary model à la [Woodford \(2003\)](#) with the crucial departure that firms are allowed to observe past aggregate prices to infer monetary shocks. When pricing complementarities are sufficiently strong, prices fail to perfectly aggregate information, and monetary shocks can generate long-lasting real effects. Moreover, the impulse response function of inflation forecasts displays an initial underreaction and a delayed overreaction, which is consistent with the evidence on expectations documented in [Angeletos, Huo, and Sastry \(2020\)](#), [Kucinskas and Peters \(2018\)](#), and [Bianchi, Ludvigson, and Ma \(2020\)](#). In contrast with the literature that resorts to certain deviation from rationality, this empirical pattern turns out to be a natural outcome in our setting with rational expectations.<sup>3</sup>

Second, we apply our results to New Keynesian models with incomplete information. We show that with dispersed information, the confounding regime is more likely to arise with a lower nominal rigidity in the New Keynesian Phillips curve and a higher marginal propensity to consume (MPC) in the dynamic IS curve. With more flexible prices, those firms that can adjust their prices need to pay more attention to others' prices; with a higher MPC, consumers are more sensitive to the aggregate demand. Both of these map to stronger GE feedback effects which manifest themselves as stronger strategic complementarities, and hence lead to non-invertible aggregate outcomes.

When the central bank employs a Taylor-type monetary policy, a more aggressive response of nominal interest rate to inflation reduces the positive feedback effects between the supply and demand block, resulting in a lower degree of strategic complementarity. With incomplete information, a more aggressive policy not only modifies the magnitude of the economy's response to shocks, but it can also alter the entire dynamic pattern by shifting the economy from the confounding to the revealing

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<sup>3</sup>For example, in [Angeletos, Huo, and Sastry \(2020\)](#), this pattern is rationalized by imposing a behavioral bias—over-extrapolation.

regime. It follows that a dovish monetary policy rule tends to induce hump-shaped and oscillatory responses of inflation and output.

## Related Literature

The study of how well endogenous variables like prices summarize relevant dispersed information dates back to [Hayek \(1945\)](#). What makes information aggregation nontrivial in our model is the fact that fundamentals and signals are correlated over time. Our work therefore complements the line of research that focuses on endogenous information aggregation that relies on the history of signals, including, for instance, [Townsend \(1983\)](#), [Sargent \(1991\)](#), [Kasa \(2000\)](#), [Nimark \(2014\)](#), and [Bacchetta and Wincoop \(2006\)](#). This paper is closely related to [Rondina \(2008\)](#), [Graham and Wright \(2010\)](#), [Baxter, Graham, and Wright \(2011\)](#), [Kasa, Walker, and Whiteman \(2014\)](#), [Han, Ma, and Mao \(2020\)](#), and [Rondina and Walker \(2020\)](#), who also explore non-invertible equilibrium dynamics in various settings. Relative to these papers, we provide a condition for equilibrium invertibility in a general class models and explore how it relates to the degree of complementarity. In addition, whereas the equilibrium dynamics in the aforementioned papers follow tractable finite-state processes,<sup>4</sup> in our paper, the dynamics is more involved and we prove it does not permit a finite-state representation.<sup>5</sup>

This paper is also related to the large literature on monetary non-neutrality due to informational frictions. [Lucas \(1972, 1973\)](#) showed this is consistent with rational expectations. In that setup, real effects of monetary shocks were predicated on variations in aggregate nominal expenditure not being forecastable. It follows that, if data on past aggregates were available these effects could, at most, be transitory. [Woodford \(2003\)](#), following [Sims \(2001\)](#), argues that the main informational bottleneck is not its plain availability but the “limited capacity of private decision-makers to pay attention” to it. Then, assuming aggregates are never fully observed, he shows that monetary shocks can have persistent effects. One could also generate persistent effects by including additional exogenous aggregate shocks in the model or imposing rational inattention, as in [Nimark \(2008\)](#), [Adam \(2007\)](#), [Maćkowiak and Wiederholt \(2009\)](#), and [Melosi \(2017\)](#), or endogenous sentiment shocks, as in [Acharya, Benhabib, and Huo \(2017\)](#). In this paper, we show that even if past aggregates are perfectly observed, monetary policy can have long-lasting effects which arise endogenously only if GE effects are strong enough.<sup>6</sup>

A common assumption in the literature is that information is static or the underlying states are perfectly revealed after one period, which allows one to focus on the within-period inference problem. [Grossman \(1976\)](#) and [Hellwig \(1980\)](#) studied to what extent prices can summarize multidimensional

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<sup>4</sup>[Baxter, Graham, and Wright \(2011\)](#) solves the model numerically and the impulse response functions in their paper do not display the oscillation pattern as in our paper.

<sup>5</sup>This is the case even for square information structures—when the number of signals is equal to the number of shocks—which usually lead to more tractable equilibria.

<sup>6</sup>An exception is [Hellwig and Venkateswaran \(2015\)](#). They identify special cases in which dispersed information is irrelevant for allocation.

information in a static setting. [Messner and Vives \(2005\)](#), [Angeletos and Pavan \(2009\)](#), [Amador and Weill \(2010, 2012\)](#) and [Vives \(2014, 2017\)](#) study learning externalities associated with the observation of endogenous aggregates. A particular type of learning externality plays an important role in deviations from perfect information aggregation in our model: strategic complementarity leads agents to respond less to their private information, which has the negative side effect of reducing the informativeness of aggregates in general and prices in particular. In persistent information settings as the one considered in our paper, this crowding out can lead to non-invertibility and oscillatory dynamics. [Angeletos, Iovino, and La'O \(2020\)](#) explore optimal policy under this type of learning externality. They find that agents are better off when incentivized to respond more aggressively to their belief variations, which helps to achieve better information aggregation. [Gaballo \(2018\)](#) and [Chahrour and Gaballo \(2019\)](#) show that learning from prices can play either a propagating or a mitigating role. Their mechanism relies on agents receiving different types of signals, while our framework focuses on the classical case with a common information structure.

The rest of the paper is organized as follows: Section 2 sets up a simple monetary model with informational frictions. Section 3 establishes the main result about the two regimes of price dynamics. Section 4 introduces noisy price observation which allows us to bridge the predictions of the model with the ones from its exogenous-information counterpart. Section 5 extends the main results to more general information and payoff structures, and explores various applications. Section 7 concludes.

## 2. A MONETARY MODEL

In this section, we introduce a simple model of monopolistic competition to illustrate the basic idea of information aggregation through endogenous actions. The model is largely borrowed from [Woodford \(2003\)](#)<sup>7</sup> and it is designed not to exhaust the implications of our main results but to be the simplest setup in which we can highlight the role of strategic complementarity in determining the information content of endogenous information. It allows us to examine the question of how efficiently prices and other macro variables aggregate the information that firms need to make their decisions.

Suppose the producer of good  $i$  chooses the current and future prices of their good,  $\{P_{it+k}\}$ , to maximize the expected present value of future (real) profits

$$\mathbb{E}_{i,t} \left[ \sum_{k=0}^{\infty} \beta^k m(Y_{t+k}) \left[ \frac{P_{it+k}}{P_{t+k}} Y_{i,t+k} - C(Y_{i,t+k}, Y_{t+k}) \right] \right]$$

subject, in every period  $t + k$ , to the demand function

$$Y_{i,t+k} = Y_{t+k} \left( \frac{P_{i,t+k}}{P_{t+k}} \right)^{-\theta}, \quad (2.1)$$

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<sup>7</sup>The main difference relative to [Woodford \(2003\)](#) is that we consider a more general information structure.

for some  $\theta > 1$ . The producer takes the Dixit-Stiglitz index of real aggregate demand,  $Y_t$ , and the corresponding price index,  $P_t$ , as given. The real cost of producing is given by  $C(Y_{it}, Y_t)$  and depends not only on the amount produced by the firm,  $Y_{it}$ , but also on the aggregate output,  $Y_t$ , to account for its effect on factor prices. The firm weights different states by the stochastic discount factor,  $m(Y_t)$ , so that the expected discounted profits can be interpreted as a financial market valuation of the firm.

To focus on informational frictions we abstract from price frictions here. The firms can choose their prices every period independent of the past. Thus, in period  $t$ , firm  $i$  chooses  $P_{it}$  to maximize

$$\mathbb{E}_{it} \left[ Y_t \left( \frac{P_{it}}{P_t} \right)^{1-\theta} - C \left( Y_t \left( \frac{P_{it}}{P_t} \right)^{-\theta}, Y_t \right) \right],$$

where  $\mathbb{E}_{it}$  denotes the expectation conditional on the information set of firm  $i$  in period  $t$ . The optimal pricing decision of firm  $i$  implies a first order condition which can be log-linearly approximated by

$$p_{it} = \mathbb{E}_{it}[(1 - \alpha)y_t + p_t],$$

where lower-case variables denote log-deviations from the full-information steady-state versions of the corresponding upper-case variables, which satisfy  $P_{it} = P_t$  and  $Y_t = Y^*$ . The parameter  $\alpha$  can be written as a function of deep parameters.<sup>8</sup>

Changes in nominal aggregate demand,  $Q_t = P_t Y_t / Y^*$ , can be decomposed into changes in the price level and in the real output,

$$q_t = p_t + y_t.$$

We assume that the nominal aggregate demand is determined exogenously by the central bank, following an AR(1) process<sup>9</sup>

$$q_t = \rho q_{t-1} + \eta_t, \quad \eta_t \sim \mathcal{N}(0, 1).$$

Therefore, though the innovations to the nominal aggregate demand,  $\eta_t$ , can have a broader interpretation, from now on, we refer to them as monetary shocks.

From an individual firm's perspective, their pricing decision can be expressed as the best-response function of a standard static beauty-contest game, that is

$$p_{it} = (1 - \alpha)\mathbb{E}_{it}[q_t] + \alpha\mathbb{E}_{it}[p_t], \quad \text{with} \quad p_t = \int p_{it} \, di, \quad (2.2)$$

and where  $\alpha \in (-1, 1)$  controls the degree of strategic complementarity between firms, or the strength of GE feedback effects—we use both terms interchangeably.

With incomplete information, the aggregate price depends not only on the firms' first-order ex-

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<sup>8</sup>More specifically,  $Y^*$  solves  $(1 - \theta) + \theta C_y(Y^*, Y^*) = 0$ , and  $\alpha \equiv 1 - \frac{C_{yy}(Y^*, Y^*)Y^* + C_{yY}(Y^*, Y^*)Y^*}{C_y(Y^*, Y^*) + \theta C_{yy}(Y^*, Y^*)}$ .

<sup>9</sup>We relax this assumption in Section 5.

pectations about the nominal aggregate demand but also on higher-order expectations as emphasized in [Morris and Shin \(2002\)](#) and [Woodford \(2003\)](#). We formalize this in the following lemma.

**Lemma 1.** *The aggregate price is given by*

$$p_t = (1 - \alpha) \sum_{k=0}^{\infty} \alpha^k \bar{\mathbb{E}}_t^{k+1}[q_t], \quad (2.3)$$

where the higher-order expectations are defined recursively as follows

$$\bar{\mathbb{E}}_t^0[q_t] \equiv q_t, \quad \text{and} \quad \bar{\mathbb{E}}_t^{k+1}[q_t] \equiv \int \mathbb{E}_{it} \left[ \bar{\mathbb{E}}_t^k[q_t] \right] di.$$

The real effects of a monetary shock on output are captured by the gap between  $p_t$  and  $q_t$ . Condition (2.3) implies that this gap depends on the dynamics of all higher-order expectations, which in turn depend on the information structure faced by firms. To proceed, we first consider a benchmark case with dispersed but *exogenous* information, in which the information content is independent of equilibrium objects. In the next section, we turn to the *endogenous* information case, in which the information content is a function of equilibrium objects.

**Exogenous Information Benchmark.** Suppose that every period,  $t$ , firm  $i$  receives a new private signal,  $x_{it}$ , about the nominal aggregate demand<sup>10</sup>

$$x_{it} = q_t + u_{it}, \quad u_{it} \sim \mathcal{N}(0, \tau^{-1}). \quad (2.4)$$

With  $\tau \rightarrow \infty$ , we return to the frictionless case where firms observe nominal aggregate demand,  $q_t$ , perfectly. Not only is there no first-order uncertainty about the fundamental, but also all higher-order expectations become common knowledge and collapse to the first-order expectation, that is,  $\bar{\mathbb{E}}_t^k[q_t] = q_t$  for all  $k \geq 0$ . It follows that

$$p_t^* = q_t, \quad y_t^* = 0. \quad (2.5)$$

Thus, prices vary with the nominal aggregate demand in a one-to-one fashion, leaving aggregate output unchanged.

With  $\tau < \infty$ , information is incomplete, and the equilibrium outcome is shaped by the complicated dynamics of all the higher-order expectations in condition (2.3). We can, however, bypass this complexity by applying the single-agent result in [Huo and Pedroni \(2020\)](#), which directly yields a

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<sup>10</sup>This information structure is similar to those used in [Woodford \(2003\)](#) and [Angeletos and La'O \(2010\)](#).

simple characterization of the price dynamics that follows an AR(2) process:<sup>11</sup>

$$p_t = \left(1 - \frac{\vartheta}{\rho}\right) \frac{1}{1 - \vartheta L} q_t, \quad (2.6)$$

where  $\vartheta \in (0, \rho)$  is given by

$$\vartheta = \frac{1}{2} \left( \rho + \frac{1 + (1 - \alpha)\tau}{\rho} - \sqrt{\left( \rho + \frac{1 + (1 - \alpha)\tau}{\rho} \right)^2 - 4} \right).$$

The aggregate price moves less than the nominal aggregate demand, captured by the term  $1 - \vartheta/\rho$  in equation (2.6), which implies that real output co-moves with its nominal counterpart. In addition, it takes time for firms to learn the monetary shock; and the deviation of prices from the nominal aggregate demand is persistent, captured by the term  $1/(1 - \vartheta L)$ . This leads to long-lasting effects of monetary shocks on real output.<sup>12</sup>

### 3. ENDOGENOUS INFORMATION AGGREGATION

In this section, we allow the signals in firms' information set to include endogenous aggregate variables, so that the informativeness of the information is determined in equilibrium. We show that this change can lead to significantly different price and output dynamics, and affects how much can be learned from observing aggregate variables.

**Information Structure.** We assume that in any period  $t$ , signals in firms' information set include

1. private exogenous signals,  $\{x_{i,t}, x_{i,t-1}, \dots\}$  as in (2.4);
2. past prices,  $\{p_{t-1}, p_{t-2}, \dots\}$ .

The first set of signals makes information dispersed as mentioned before, while the second set of signals allow agents to better coordinate and learn from each others' actions.

There are two main reasons why we assume agents do not observe the contemporaneous aggregate outcomes but rather their delayed values: First, from a practical point of view, firms and households typically need to plan their production or purchasing with a period of time in advance of the realizations the fundamentals like demand shocks or income. There are lags in the publication of aggregate statistics as well, and nowcasts are necessary for most aggregate variables probably with the exception of frequently traded asset prices. Second, from a theoretical point of view, the simultaneity of making

<sup>11</sup>This price dynamics is similar to that obtained in [Rondina \(2008\)](#) where individuals' best responses depend on idiosyncratic fundamentals instead a common one.

<sup>12</sup>In [Angeletos and Huo \(2018\)](#), aggregate outcomes follow similar dynamics in a forward-looking game, where the determination of  $\vartheta$  is more complicated.



an individual decision while observing the aggregate outcomes is easier to be justified when the aggregate states are perfectly observable and agents can form rational expectations about the aggregate variables without frictions. In our economy with dispersed information, observing past outcomes is a more natural timing of decisions as aggregate states are observed with noise. Nevertheless, the period length is a priori not determined, so one could consider the smallest of lags and the analysis that follows still applies.

To measure the amount of information aggregated by price, we define price informativeness as

$$\chi \equiv 1 - \frac{\text{Var}[q_t - \mathbb{E}[q_t|p^t]]}{\text{Var}[q_t]},$$

where  $p^t$  denotes the history of price observations  $\{p_t, p_{t-1}, \dots\}$ . Notice that  $\chi \in [0, 1]$ . When prices perfectly reveal the nominal aggregate demand, the mean squared prediction error is zero and price informativeness reaches its maximum level,  $\chi = 1$ . When prices are not informative at all, the forecast  $\mathbb{E}[q_t|p^t] = 0$  and  $\chi = 0$ .

### 3.1 Invertibility

We start with a thought experiment in which only a single firm can observe past prices. As this firm is infinitesimal, the aggregate price process is the same as in the exogenous information case. How much can this firm learn from the history of prices? To answer this question, we rearrange condition (2.6), which leads to

$$q_{t-1} = \left(1 - \frac{\vartheta}{\rho}\right)^{-1} (p_{t-1} - \vartheta p_{t-2}). \quad (3.1)$$

That is, by observing the history of prices, the uncertainty about *past* shocks is completely eliminated. Price informativeness is, then, at its maximal level,  $\chi = 1$ .

This is the basic logic behind the conventional wisdom that prices can effectively aggregate information. In the context of monetary neutrality, it would imply that when firms observe past prices, the monetary shock can only have transitory effects. However, this logic does not necessarily extend to the case in which all firms can observe past prices, since the different information structure leads to different equilibrium price dynamics. Importantly, the mapping from prices to underlying shocks obtained in condition (3.1) may not be possible. What determines if a price process is able to perfectly aggregate information is its invertibility, as we formalize in the next lemma.

**Lemma 2.** A price process  $p_t = g(L)\eta_t$  is invertible if  $g(L)$  does not contain any inside root.<sup>13</sup> When invertible, prices perfectly aggregate information about the underlying shock, i.e.,

$$\mathbb{E}[q_{t-k}|p^t] = \frac{g(L)^{-1}p_{t-k}}{1 - \rho L} = q_{t-k}, \quad \text{for } k \geq 0. \quad (3.2)$$

**Corollary 1.** With exogenous information, for all admissible parameters  $(\alpha, \tau, \rho)$ , the price process is always invertible and the history of prices contains the same information.

The aforementioned exogenous information case is an example of an invertible process. Perhaps somewhat surprisingly, even with very large noise (small  $\tau$ ), the history of prices still reveals all the information about the monetary shock, even though the magnitude of the price response can be arbitrarily small.

### 3.2 Two Regimes of Information Aggregation

When information is endogenous and dispersed, the extent to which prices efficiently aggregate information hinges both on the severity of informational frictions and, crucially, on the degree of strategic complementarity.

**Proposition 3.1.** The equilibrium price perfectly aggregates information, that is,  $\mathbb{E}[q_t|p^t] = q_t$ , if and only if the triple  $(\alpha, \tau, \rho)$  is in the invertible region that satisfies

$$\alpha < 1 - \frac{\rho}{\tau}. \quad (3.3)$$

*Proof.* See Appendix A. □

This proposition partitions the parameter space into two regions: the invertible region in which prices fully reveal the underlying fundamental, and the non-invertible region in which they do not. In the invertible region,  $p^t$  contains the same information as the nominal aggregate demand  $q^t$ , and therefore we call this the *perfect revealing regime*. In the non-invertible region,  $p^t$  contains less information than  $q^t$  and firms are no longer able to infer the monetary shocks perfectly, which we refer to as the *confounding regime*. Note that the invertibility condition used here is similar to the notion of informativeness in [Rondina \(2008\)](#). If we instead consider an independent-value best response, we would recover the same condition obtained there.

As shown in [Figure 1](#), the invertible region features relatively mild degrees of informational frictions and strategic complementarity. The opposite prevails in the non-invertible region. To understand this partition of the parameter space, we first go through a more “mechanical” explanation, and then we provide a more intuitive argument. Consider the impulse response function of prices

<sup>13</sup>The function  $g(L)$  is said to have an inside root if there exists a complex number,  $z$ , inside the unit circle of the complex plane, such that  $g(z) = 0$ .

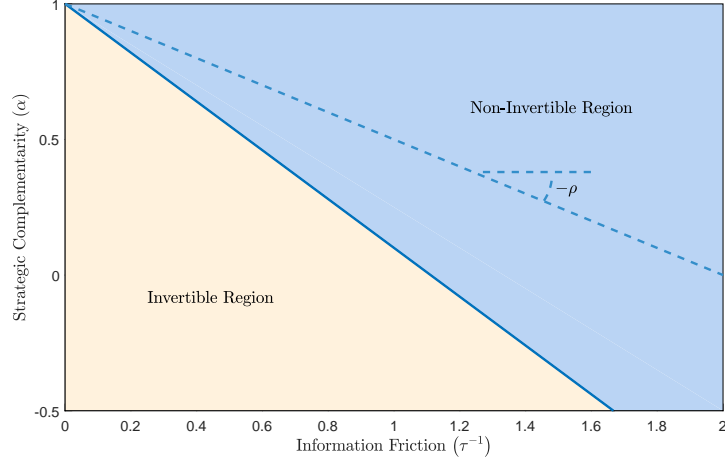


Figure 1: Two Regions of Price Informativeness

The solid line corresponds to  $\rho = 0.9$  and the dashed line to  $\rho = 0.5$ .

to a monetary shock, and suppose that the process of  $p_t$  is invertible. We now explore under which conditions this conjecture can be verified. In period  $t = 0$ , firms observe the price history except for the current one. Since it is common knowledge that the monetary shocks in the past are equal to zero, to infer the current monetary shock firms rely solely on their private signal,  $x_{i,0}$ . The aggregate price level in this period,  $p_0$ , is determined the same way as in a static beauty-contest game with exogenous information. In period  $t = 1$ , firms observe  $p_0$  and, by invertibility,  $q_0$  becomes common knowledge. It follows that, in periods  $t \geq 1$ , prices are perfectly in line with the nominal aggregate demand,  $p_t = q_t = \rho^t q_0$ . Taking stock, the law of motion of price is given by

$$g(L) = \underbrace{p_0}_{\text{impact effect}} + \frac{\rho L}{1 - \rho L}, \quad (3.4)$$

where the first term,  $p_0$ , captures the impact effect, and the second term captures the response in the following periods when the monetary shock effectively becomes public.

The analysis above is based on the assumption that the process of  $p_t$  is invertible. To examine whether this is indeed the case, we need to check if any of the roots of  $g(L)$  are inside the unit circle, which is true if and only if

$$p_0 = \frac{(1 - \alpha)\tau}{1 + (1 - \alpha)\tau} > \frac{\rho}{1 + \rho}. \quad (3.5)$$

That is, prices perfectly aggregate information if and only if the initial response is large enough. It is exactly when the informational friction is relatively small (large  $\tau$ ) and the coordination motive is relatively weak (small  $\alpha$ ) that the impact response is sizable. The former increases the responsiveness of first- and higher-order expectations, and the latter decreases the weight on generally more inertial higher-order expectations.

Intuitively, the information contained in prices ultimately comes from each individual firm's

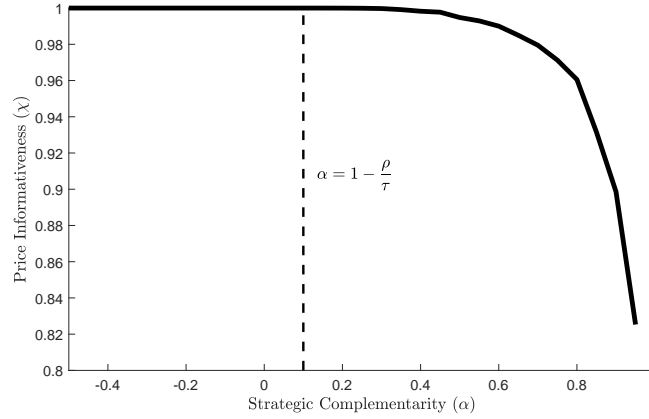


Figure 2: Price Informativeness

Parameters:  $\rho = 0.9$ , and  $\tau = 1$ .

private information. The more firms rely on their private signals when making decisions, the more information can be potentially encoded in prices. With more private signal noise or higher degree of complementarity, firms put more weight on information in the public domain which reduces the usage of private signals. This intuition is reminiscent of the one in [Amador and Weill \(2010\)](#), where an increase of the public signal's precision may actually lower the information content of prices as the information contained in private signals is utilized less than a social planner would prefer.

This intuition is subject to the following caveat. Recall that, with exogenous information, independent of the informational friction and coordination motive, the price process is always invertible. With endogenous information, invertibility is obtained in a subset of the parameter space. This difference highlights that the informativeness of prices is affected not only by their overall responsiveness but also by their particular dynamic pattern. The importance of *dynamic* information aggregation also manifests itself via condition (3.3): with  $\rho = 0$ , information is always perfectly aggregated. Moreover, a higher  $\rho$  moves the fundamental process away from a static i.i.d. and increases the non-invertible region, as can be seen in Figure 1.

Proposition 3.1 already indicates that varying the degree of strategic complementarity affects the invertibility of prices. Figure 2 shows how price informativeness changes with the complementarity. A higher  $\alpha$  diverts agents from the use of private information, which tends to lower the price informativeness. In the non-invertible region, this is indeed the case. [Woodford \(2003\)](#) emphasizes that higher strategic complementarity results in more inertia in inflation as higher-order expectations play a more important role. Our results imply that, when information is endogenous, there is an additional information channel through which the degree of complementarity shapes the aggregate outcome, a point we further elaborate on in Section 4.

### 3.3 Price Dynamics

We have characterized the invertible region in which prices perfectly aggregate information and the corresponding price dynamics. What does the stochastic process for prices look like if the parameters are in the non-invertible region? The following proposition shows that the price dynamics become significantly more complex.

- Proposition 3.2.** 1. *In the invertible region, there exists a unique law of motion of  $p_t$ :  $p_t = q_t - \frac{1}{1+(1-\alpha)\tau} \eta_t$ .*
2. *In the non-invertible region, the law of motion of  $p_t$  does **not** have a finite-state representation.*

*Proof.* See Appendix B. □

In the invertible region, the law of motion for prices differs from the nominal aggregate demand only at the impact response, following an ARMA(1,1) process. To visualize the process, the red dashed line in Figure 3 plots the impulse response of prices,  $p_t$ , with  $\alpha = 0.1$ , which is on the edge of the invertible region (we set  $\rho = 0.9$ , and  $\tau = 1$ ). Except for the first period, prices are equal to nominal aggregate demand. Varying the parameters within the invertible region only changes the initial response. Recall that real output is equal to the difference between the nominal aggregate demand and the price level,  $y_t = q_t - p_t$ . So, in this case, the effect of the shock on real output is only transitory. This is in line with the conventional wisdom that the effects of a monetary shock to real aggregates are transitory when prices become public.

In the non-invertible region, the law of motion for  $p_t$  does not follow any finite ARMA process. In the literature, when the number of signals is the same as the number of shocks, one can typically obtain the equilibrium law of motion with a finite-state representation (see [Kasa \(2000\)](#), [Kasa, Walker, and Whiteman \(2014\)](#), [Rondina and Walker \(2020\)](#) for example). In contrast, in our environment, where the fundamental is invertible and the non-invertibility of equilibrium prices is obtained endogenously when the degree of strategic complementarity is high enough, no stochastic process with a finite number of inside roots can be supported as an equilibrium. We obtain this result by a proof of contradiction: assuming the equilibrium process has a finite number of inside roots, we show that the implied actual outcome always contains more inside roots than initially conjectured. Unlike the invertible region, in the non-invertible region we cannot guarantee that there exists a unique equilibrium.<sup>14</sup> However, our results hold for all possible equilibria that may arise in the non-invertible region. For the parameters considered in this paper, our numerical solution always converges to the same equilibrium.

In Figure 3, the blue solid line shows the impulse response for  $p_t$  with  $\alpha = 0.9$ , which is in the non-invertible region. The response of  $p_t$  is initially dulled, and builds up gradually. Different from the invertible case, it does not coincide with the nominal demand,  $q_t$ , after period  $t = 1$ . Instead,

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<sup>14</sup>To the best of our knowledge, the literature has not been able to give a general proof of the uniqueness result when information is endogenous and persistent.

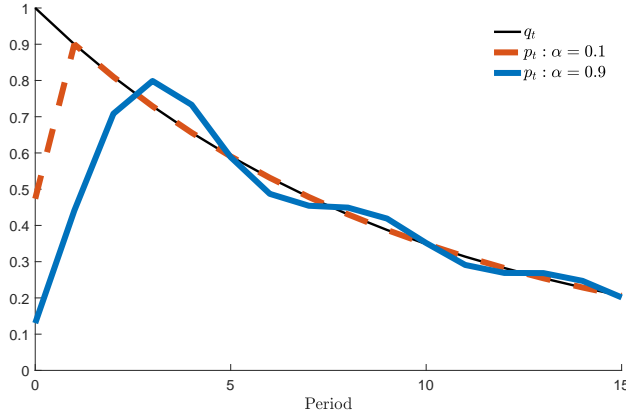


Figure 3: Response of Aggregate Price to Monetary Shock

Parameters:  $\rho = 0.9$ , and  $\tau = 1$ .

it oscillates around  $q_t$ . This pattern is the result of the imperfect dynamic information aggregation. To understand this oscillation pattern, it is useful to consider the following two observations: First, recall from Lemma 2 that prices are essentially the average forecasts about the fundamental, which cannot systematically deviate from the nominal demand. For example, with exogenous information, the impulse response of price is uniformly below that of the nominal aggregate demand, and this implies a persistent underestimation of the monetary shock and past prices. However, this type of persistent underestimation of past prices does not square with the fact that past prices are already part of agents' information set. Second, prices cannot be precisely equal to the nominal aggregate demand, as this would make them perfectly informative, contradicting the assumed non-invertibility of the price process. As a result, the only possible scenario is the waves of optimism and pessimism that we see here.<sup>15</sup>

A direct implication of Proposition 3.2 is that the monetary shocks can induce persistent real output fluctuations. To allow long-lasting effects of monetary shocks on output, it is necessary that prices do not fully reveal the underlying shock. Woodford (2003) directly assumes that past prices cannot be observed, which can be rationalized by rational inattention, as in Maćkowiak and Wiederholt (2009). Another way to avoid perfect revelation is to add additional exogenous aggregate shocks, as in Nimark (2008) and Melosi (2017), or to add endogenous sentiment shocks, as in Acharya, Benhabib, and Huo (2017). Our results differ from those in the literature. The non-invertibility of the price process is an endogenous equilibrium outcome, and whether it is indeed the case depends on the underlying parameters. Prices efficiently aggregate information only when individual firms' responses to their *private* signals are sufficiently strong. When such response is relatively weak, monetary shocks have persistent real effects even if past prices are perfectly observed.

We conclude this section with a remark on uniqueness of equilibrium. In general, multiple

<sup>15</sup>Due to the fact that the price process does not allow a finite-state representation, the equilibrium can only be solved numerically. We impose the price process follows a finite MA process to approximate the actual price process, and we choose the finite MA process to minimize the distance between the perceived and the actual price process.

equilibria may arise when information is endogenous (see [Gaballo \(2018\)](#) and [Acharya, Benhabib, and Huo \(2017\)](#) for example). In our setting, the equilibrium is unique in the invertible region, which we show by construction. Though we cannot rule out multiplicity in the non-invertible region, we think it is not likely for the following reason: the common cause of multiplicity is the strong complementarity between the responsiveness of the current action and the informativeness of endogenous signals. Since we assume that the aggregate outcome is observed with a one-period delay, the informativeness of newly arrived signals does not depend on agents' contemporaneous actions.<sup>16</sup> Lastly, regardless of whether the equilibrium is unique or not, all the theoretical results and analyses we have derived remain true.

## 4. IMPERFECT PRICE OBSERVATION

In this subsection, we explore to what extent previous results remain true when prices are not perfectly observed. The imperfect observation could arise when prices are recorded with measurement errors, or firms perceive published prices in an imperfect way due to costly contemplation. We show that the exogenous information case in Section 2 and the endogenous information case with perfect price observation in Section 3 serve as two useful benchmarks: the outcomes of the noisy price observation case live in the middle of these two extremes. In addition, we suggest a potential empirical footprint for endogenous information aggregation.

### 4.1 Noisy Prices: An Intermediate Case

Suppose firms observe past prices with idiosyncratic noise  $v_{it}$ ,

$$z_{it} = p_{t-1} + v_{it}, \quad v_{it} \sim \mathcal{N}(0, \kappa^{-1}). \quad (4.1)$$

The information set for firm  $i$ , then, becomes  $\mathcal{I}_{it} = \{x_i^t, z_i^t\}$ . This information structure nests the two benchmarks we have studied: by setting  $\kappa$  to infinity, we return to the case where prices are perfectly observed; by setting  $\kappa$  to zero, we obtain the other extreme where firms only have access to exogenous information,  $\{x_i^t\}$ . In the intermediate range, information is still endogenous as the informativeness of the price signal,  $z_{it}$ , is determined endogenously in equilibrium.<sup>17</sup>

We first show how the price dynamics changes when we vary the noise to the price observation. In Panel (b) of Figure 4, the blue solid line corresponds to the case in which prices are perfectly observed,  $\kappa \rightarrow \infty$ , and the parameters lie in the non-invertible region. When the magnitude of the private noise is positive but relatively small, by continuity, the price dynamics is similar to the limiting case. The

<sup>16</sup>If there is an initial period, it follows that the equilibrium is always unique no matter where it converges to. Moreover, though this is certainly not definitive, our numerical algorithm always yields a unique solution.

<sup>17</sup>To solve the intermediate case, one has to resort to numerical methods as, generally, no finite-state solution is available. See [Huo and Takayama \(2017\)](#) for a more detailed discussion.

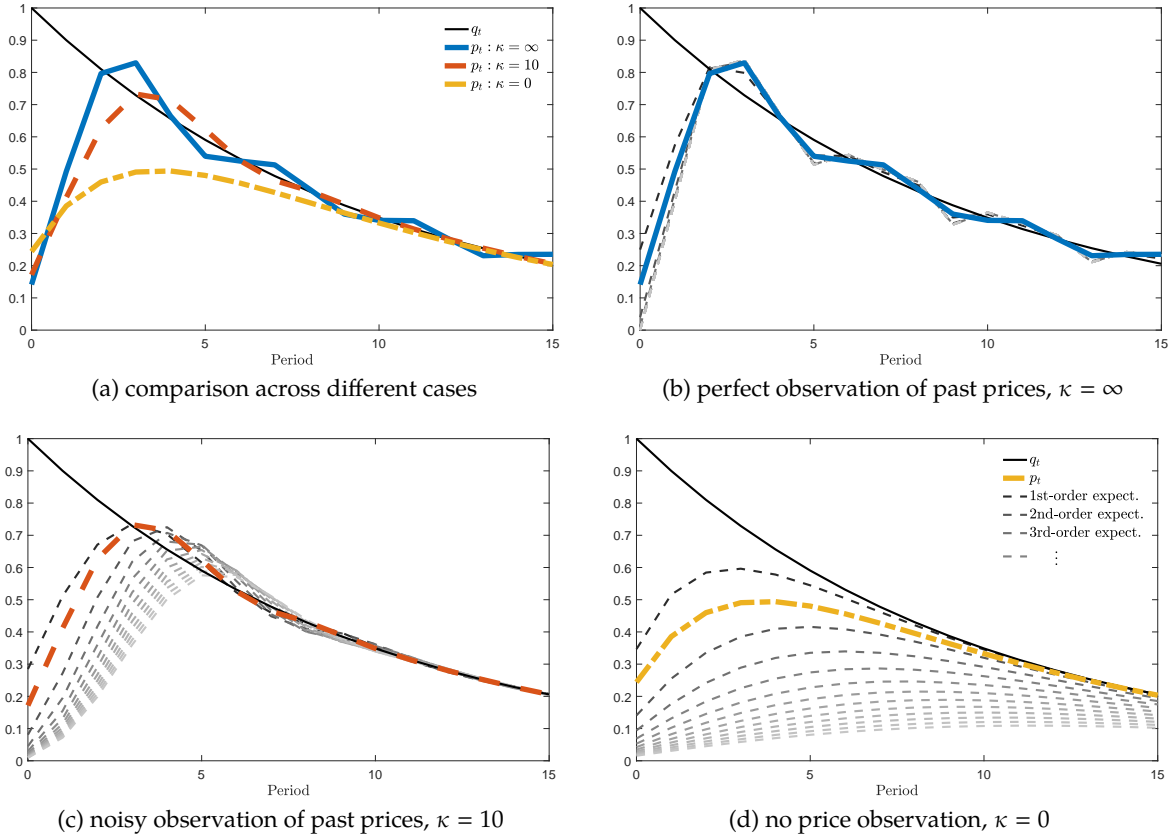


Figure 4: Response of Aggregate Prices Varying Precision of Price Observation ( $\kappa$ )

Parameters:  $\rho = 0.9$ ,  $\alpha = 0.5$ , and  $\tau = 0.25$ .

red dashed line in Panel (c) confirms this logic: with  $\kappa = 10$ , it resembles the pattern of the blue solid line though the oscillation around the fundamental is less pronounced. When  $\kappa = 0$  (the exogenous information case), there is no oscillation anymore. The yellow line in Panel (d) is always below the fundamental. In summary, our analysis in the previous two sections can serve as two baseline cases. As  $\kappa$  changes from infinity to zero, the properties of the price dynamics shift from one extreme to the other.

A notable implication of dispersed information is that higher-order expectations differ from first-order expectations. Depending on the information structure, higher-order expectations display quite different patterns, as shown in Panels (b) to (d) in Figure 4. With only private exogenous information, there is little information in the public domain besides the common prior. As a result, higher-order expectations significantly depart from first-order expectations. On the other end of the spectrum, with perfect observation of past prices, the expectations about the fundamental are much better coordinated, resulting in a high level of synchronization among first-order and higher-order expectations. With moderate private noise on the price observation, the patterns of higher-order expectations again strike a balance between the two benchmarks.



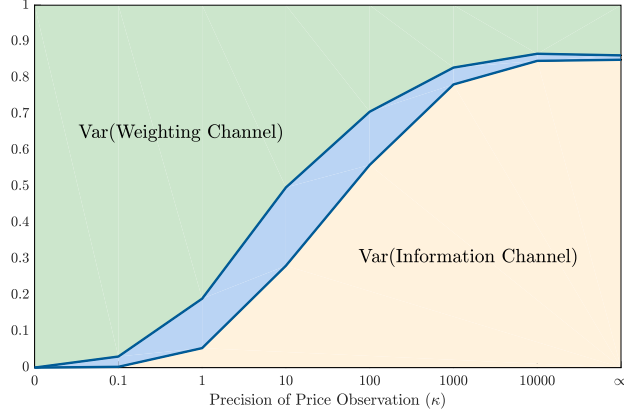


Figure 5: Impact of Changing  $\alpha$  from 0 to 0.5.

Note: The figure displays a variance decomposition of the two terms on the right-hand side of equation (4.2), with the blue region in the middle corresponding to the covariance term.

The comparison above implies that the way in which the strategic complementarity shapes the outcomes hinges on whether information is endogenous or not. In particular, the effect of shifting the degree of complementarity from  $\alpha$  to  $\hat{\alpha}$  can be split into two parts,

$$p_t^{\hat{\alpha}} - p_t^{\alpha} = (1 - \hat{\alpha}) \sum_{k=0}^{\infty} \hat{\alpha}^k \{ \mathbb{E}_t^{k+1}[q_t; \hat{\alpha}] - \mathbb{E}_t^{k+1}[q_t; \alpha] \} + \sum_{k=0}^{\infty} \{ (1 - \hat{\alpha})\hat{\alpha}^k - (1 - \alpha)\alpha^k \} \mathbb{E}_t^{k+1}[q_t; \alpha]. \quad (4.2)$$

The first channel is associated with changes in higher-order expectations, we call this the information channel. A different  $\alpha$  may modify the informativeness of prices which, in turn, affects higher-order expectations. The second channel is the weighting channel. It has to do only with the fact that, for higher levels of  $\alpha$ , the equilibrium actions put relatively more weight on expectations of higher order. With exogenous information, only the second channel is at work. With perfect price observation, though qualitatively both channels are present, the impact of the second channel is quantitatively limited as higher-order expectations are close to first-order expectations. Figure 5 displays the variance decomposition of prices,  $p_t$ , into the information and weighting channels. With noisy price observation, both channels can have similar quantitative contributions.

## 4.2 Delayed Overreaction in Beliefs

In this subsection, we connect the dynamics of expectations under endogenous information with its empirical counterpart. [Angeletos, Huo, and Sastry \(2020\)](#) recently showed that forecasts about inflation display underreaction in the short run which is then followed by a delayed overreaction. This oscillatory pattern of forecast errors, which is also identified in [Kucinskas and Peters \(2018\)](#), could help reconcile the seemingly conflicting evidence documented by [Coibion and Gorodnichenko \(2015\)](#) and [Kohlhas and Walther \(2019\)](#)—the former finds underreaction of average forecasts while

the latter finds overreaction. [Angeletos, Huo, and Sastry \(2020\)](#) rationalize this pattern by combining dispersed information with over-extrapolation.<sup>18</sup> Without introducing distorted beliefs, this pattern can be a natural result with endogenous information aggregation under rational expectations.

To show this, we extend the baseline model in Section 2 in two ways: First, we allow the observation of past prices with noise as in equation (4.1). Second, for comparability, we allow the growth rate of the nominal aggregate demand to follow an AR(1) process as in [Woodford \(2003\)](#),

$$q_t - q_{t-1} = \rho(q_{t-1} - q_{t-2}) + \eta_t. \quad (4.3)$$

To explore the effects of the introduction endogenous information on equilibrium outcomes, Figure 6 shows the impulse responses for output, inflation, and inflation forecast errors for two versions of the model. The left panel shows what happens with dispersed but *exogenous* information. It reproduces the last panel in Figure 5 of [Woodford \(2003\)](#), which he uses to argue that the model can generate a hump-shaped response of inflation and output with a delayed peak for inflation. Turning to the case with endogenous information, which is shown in the right panel. We maintain the same parametrization as in [Woodford \(2003\)](#), but allow agents to imperfectly observe past prices as in equation (4.1), with precision  $\kappa = 1$ .<sup>19</sup> In this case, besides inflation peaking after output, both also display an oscillatory time pattern.

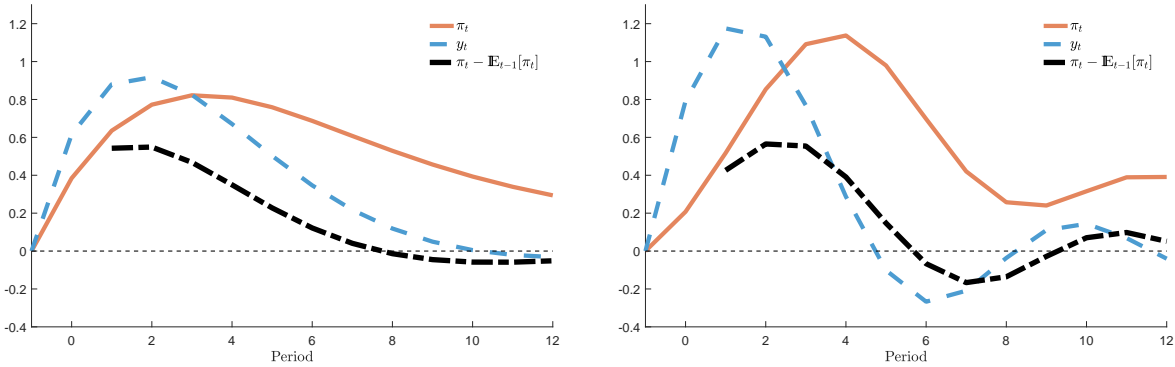


Figure 6: Comparison of impulse response functions predicted by the two models, for  $\kappa = 0$  (Exogenous Information) and  $\kappa = 1$  (Endogenous Information)

Other parameters as in [Woodford \(2003\)](#):  $\rho = 0.9$ ,  $\alpha = 0.85$ , and  $\tau = 1/6$ .

Returning to the discussion of the time pattern of forecast errors, notice that both models are able to generate the initial underreaction of inflation forecasts. The black dashed curves in Figure 6 plot the one-period ahead inflation forecast errors, and the fact that they are positive initially indicates this underreaction. Inspecting the results on the left panel, one can see that the exogenous information

<sup>18</sup>Within the realm of rational expectations, [Kohlhas and Walther \(2019\)](#) shows that by allowing asymmetric attention to different components of the outcome, unconditionally, the forecasts can display overreaction to current outcomes and underreaction to forecast revisions. While in [Angeletos, Huo, and Sastry \(2020\)](#), the identified dynamic pattern of forecasts is the response conditional on a particular shock.

<sup>19</sup>If we allow agents to perfectly observe past prices, the equilibrium is non-invertible.

model is actually able to generate small amounts of delayed overreaction. However, the delayed overreaction is significantly magnified by the introduction of endogenous learning as the forecast errors inherit the oscillatory pattern from inflation itself. Further, even higher amounts of delayed overreaction can be obtained by adjusting the precision of the price observation,  $\kappa$ , which here has been set arbitrarily to  $\kappa = 1$ . We think, therefore, that the model offers a promising explanation for this type of dynamics.<sup>20</sup>

Bordalo, Gennaioli, Ma, and Shleifer (2020) examine the correlation between forecast errors and forecast revisions for various variables at the *individual* level, and show that agents tend to overreact to news on average. Our benchmark model could not replicate this fact, since some deviation from rational expectations is required. Nevertheless, augmented the model with either overconfidence as in Broer and Kohlhas (2019) or diagnostic expectations as in Bordalo, Gennaioli, Ma, and Shleifer (2020), would not change our main results on aggregate dynamics and would help to also match the empirical evidence at the individual level.<sup>21</sup>

## 5. EXTENSIONS AND OTHER APPLICATIONS

In this section we show that the main insights developed earlier extend to significantly more involved information structures, allowing for fundamentals that can follow any stochastic process, and multiple public and private signals with noise that also follows arbitrary processes. We also extend the results to environments with more sophisticated linkages among agents featuring forward and backward complementarities, and multivariate systems that can be viewed as a network game with incomplete information.

### 5.1 General Information Structure

To facilitate the analysis, we switch to a more general notation, since the applications of the following results encompass a wide range of settings that may differ from the one considered in the monetary model analyzed above.

**Best Response.** Denote agent  $i$ 's action in period  $t$  by  $a_{it}$ . Their best response function is given by

$$a_{it} = \varphi \mathbb{E}_{it} [\xi_{it}] + \alpha \mathbb{E}_{it} [a_t], \quad (5.1)$$

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<sup>20</sup>Note that in Angeletos, Huo, and Sastry (2020), the delayed overreaction of beliefs can be observed by directly inspecting the term structure of individual's forecasts at different horizons without estimating a VAR.

<sup>21</sup>If, for instance, agents' perceived precision,  $\hat{\tau}$ , was higher than the actual precision,  $\tau$ , our results on aggregate dynamics remain true, with the only exception of replacing  $\tau$  by  $\hat{\tau}$ . However, at the individual level, econometricians would find agents overreact to their signals.

where  $\xi_{it}$  denotes the individual fundamental, which can depend in a flexible way on the aggregate and idiosyncratic shocks,  $\boldsymbol{\eta}_t$  and  $\mathbf{u}_{it}$ ,

$$\xi_{it} = \mathbf{d}(L)\boldsymbol{\eta}_t + \mathbf{e}(L)\mathbf{u}_{it}, \quad \text{with} \quad \boldsymbol{\eta}_t \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_\eta^2), \quad \text{and} \quad \mathbf{u}_{it} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_u^2).$$

The lag-operator polynomial vectors,  $\mathbf{d}(L)$  and  $\mathbf{e}(L)$ , are assumed to have square-summable coefficients. We also assume that  $\mathbf{d}(L)$  is not a constant vector to rule out fundamentals that are i.i.d. on common shocks; in which case the equilibrium is always invertible, and we rule out redundant shocks by assuming that  $\boldsymbol{\Sigma}_\eta$  and  $\boldsymbol{\Sigma}_u$  have full rank.

**Information Structure.** Agents have perfect recall and, in each period  $t$ , observe three sets of signals: (1) the previous period aggregate action,  $a_{t-1}$ , (2) a vector of private signals,  $\mathbf{x}_{it}$ , (3) and a vector of public signals,  $\mathbf{z}_t$ , where

$$\mathbf{x}_{it} = \mathbf{A}(L)\boldsymbol{\eta}_t + \mathbf{B}(L)\mathbf{u}_{it}, \quad \text{and} \quad \mathbf{z}_t = \mathbf{C}(L)\boldsymbol{\eta}_t.$$

We make two assumptions: first, that there are as many common shocks as there are public signals (including  $a_{t-1}$ ), that is  $\dim(\boldsymbol{\eta}_t) = \dim(\mathbf{z}_t) + 1$ ; second, that the matrix  $\mathbf{B}(L)$  is invertible. To see why we impose the first assumption, note that, when  $\dim(\boldsymbol{\eta}_t) < \dim(\mathbf{z}_t) + 1$ , there are more public signals than common shocks and past aggregate fundamentals are always perfectly revealed independently of the degree of strategic complementarity or of informational friction; when  $\dim(\boldsymbol{\eta}_t) > \dim(\mathbf{z}_t) + 1$ , there are more common shocks than public signals and the system is non-invertible by construction, as in Section 4. The interesting case for a discussion about invertibility is the one in which  $\dim(\boldsymbol{\eta}_t) = \dim(\mathbf{z}_t) + 1$ . The second assumption is more standard, it essentially excludes the cases in which non-invertibility is due to exogenously imposed shock processes. If  $\mathbf{B}(L) = \mathbf{B}_0$ , which is usually the case in most information structures considered in the literature, then the invertibility of  $\mathbf{B}(L)$  is only violated if  $\mathbf{B}_0$  is singular or not square in which case one of the private signals is redundant. Moreover, the polynomial matrices,  $\mathbf{A}(L)$ ,  $\mathbf{B}(L)$  and  $\mathbf{C}(L)$ , must have square-summable coefficients.

**Invertibility.** In the economy studied in Section 3 there is only one aggregate shock, the monetary shock, and invertibility is obtained if and only if the equilibrium process for the price index does not contain an inside root. When there are multiple aggregate shocks, the equilibrium process can be expressed as

$$a_t = \mathbf{g}(L)\boldsymbol{\eta}_t.$$

Driven by more than one aggregate shock, the aggregate outcome by itself can no longer reveal all underlying states. The relevant question becomes whether the history of signals, taken altogether, contains sufficient information.

Formally, we define an equilibrium process to be invertible if the history of the public signals,

$\{a^t, z^t\}$ , contains the same information as the common shocks. The following lemma provides the corresponding criterion for invertibility.

**Lemma 3.** *If  $\det \begin{bmatrix} g(L) & C(L) \end{bmatrix}'$  does not contain any inside root, the equilibrium is invertible. Then, the public signals and the aggregate outcomes perfectly aggregate information,*

$$\mathbb{E} [\eta_t | a^t, z^t] = \eta_t, \quad \text{and} \quad \mathbb{E} [\xi_{it} | a^t, z^t, x_i^t] = \xi_{it}. \quad (5.2)$$

This result generalizes Lemma 2 to a multivariate system. The exogenously specified signal structure and the endogenously determined equilibrium process jointly determine whether agents can perfectly infer past shocks in the economy. Note that, once aggregate shocks are known, idiosyncratic shocks are also known since  $\mathbf{B}(L)$  has been assumed to be invertible.

With the general information structure, we can no longer provide a simple partition of the parameter space into invertible and non-invertible regions as in Proposition 3.1. However, the basic insight derived in Section 3 remains true.

**Theorem 1.** *There exists  $\bar{\alpha} \in (-1, 1)$  such that, if  $\alpha > \bar{\alpha}$ , the equilibrium is not invertible.*

*Proof.* See Appendix C. □

The exact threshold  $\bar{\alpha}$ , above which the equilibrium is not invertible, depends on the details of the information structure. Independent of these details, however, such a threshold always exists. As in Section 3, if the degree of strategic complementarity is high enough, the equilibrium is not invertible. Under this more general information structure, however, the aggregate action may contain information on the aggregate fundamental as well as the common noise. The degree of strategic complementarity affects the invertibility of the joint dynamics of aggregate actions and public signals.

There is a sense in which the information structure we set up in this section is too general. In the following example, we consider a simplified structure that encompasses many used in the literature. Directly, it allows for the introduction of public signals to the monetary model from Section 3. We also allow agents to have idiosyncratic fundamentals which they observe with arbitrary precision. Hence, it also encompasses the information structure in the business-cycle model from Angeletos and La'O (2010) (with the addition of the observation of past aggregate outcomes). With this simpler structure, we can characterize in more detail how public signals affect information aggregation with takeaways that are still broadly applicable.

**Example: how public signals affect invertibility.** Suppose the best response function is given by equation (5.1) with the individual fundamental,  $\xi_{it}$ , and the aggregate fundamental,  $\xi_t$ , satisfying

$$\xi_{it} = \xi_t + \omega_{it}, \quad \omega_{it} \sim \mathcal{N}(0, \tau_\omega^{-1}), \quad \text{and} \quad \xi_t = d(L)\eta_t, \quad \eta_t \sim \mathcal{N}(0, 1),$$

for some arbitrary  $d(L)$  with the normalization  $d_0 = 1$ . For simplicity we set  $\varphi = 1 - \alpha$ . Every period, agent  $i$  observes last period's aggregate action,  $a_{t-1}$ , a private, and a public signal,

$$x_{it} = \xi_{it} + u_{it}, \quad u_{it} \sim \mathcal{N}(0, \tau_u^{-1}), \quad \text{and} \quad z_t = \xi_t + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, \tau_\varepsilon^{-1}).$$

In this setup, we can establish the following proposition which provides an explicit condition for invertibility. The proof of it also serves as a sketch of the proof of Theorem 1.

**Proposition 5.1.** *The equilibrium is invertible if and only if every root of the function  $\Gamma(z)$*

$$\Gamma(z) \equiv d(z) - \frac{\alpha\tau_u + \tau_\omega}{\tau_u + \tau_\omega + (1 - \alpha)\tau_u(\tau_\varepsilon + \tau_\omega)} \quad (5.3)$$

lies outside the unit circle, which is not the case for  $\alpha$  high enough.

In particular, if the fundamental follows an AR(1) process, i.e.  $d(L) = 1/(1 - \rho L)$ , it is necessary and sufficient for invertibility that

$$\alpha < 1 - \frac{\rho(\tau_u + \tau_\omega)}{\tau_u(1 + \rho + \tau_\varepsilon + \tau_\omega)}. \quad (5.4)$$

*Proof.* Suppose the equilibrium is invertible. Then, agents can infer past aggregate shocks perfectly and the effect of common shocks on the aggregate outcome can only be transitory. It follows that, the impulse response of the aggregate outcome to any common shock only differs from that of the aggregate fundamental,  $\xi_t$ , on impact. Accordingly, the law of motion of the aggregate outcome can be expressed as

$$a_t = \underbrace{g_\eta \eta_t + g_\varepsilon \varepsilon_t}_{\text{impact effect}} + (d(L) - 1)\eta_t.$$

To verify if the equilibrium is indeed invertible, we need to check the condition in Lemma 3, which, in this case, reduces to checking if any of the roots of

$$\Gamma(z) = d(z) - \frac{1 - g_\eta}{1 - g_\varepsilon}, \quad (5.5)$$

are inside the unit circle. The impact effects

$$g_\eta = \frac{(1 - \alpha)\tau_u(1 + \tau_\omega) + \tau_\varepsilon(\tau_\omega + \tau_u)}{(1 - \alpha)\tau_u\tau_\omega + (1 + \tau_\varepsilon)(\tau_u + \tau_\omega)}, \quad \text{and} \quad g_\varepsilon = \frac{\tau_\varepsilon(\tau_\omega + \alpha\tau_u)}{(1 - \alpha)\tau_u\tau_\omega + (1 + \tau_\varepsilon)(\tau_u + \tau_\omega)}, \quad (5.6)$$

can be obtained by solving a simple static forecasting problem. Then, equation (5.3) follows from substituting equation (5.6) into (5.5). Next notice that, as  $\alpha$  increases towards 1, equation (5.3) converges to  $d(z) - 1$  which has root zero, since  $d_0 = 1$ . Hence, since the roots of a polynomial are a continuous function of its coefficients, there exists some  $\bar{\alpha} \in (-1, 1)$  such that, for all  $\alpha > \bar{\alpha}$ , this equation has an inside root and the equilibrium cannot be invertible. Moreover, condition (5.4) follows immediately from solving for the root of equation (5.3) and requiring it to be outside the unit

circle. □

**Two opposing forces.** How does the precision of the public signal,  $\tau_\varepsilon$ , affect invertibility? Note that in the proof of Proposition 5.1, the root of evil that causes non-invertibility is that  $\frac{1-g_\eta}{1-g_\varepsilon} \rightarrow 0$  when the magnitude of  $\alpha$  approaches to zero. In general, the equilibrium is more likely to be invertible, or  $\Gamma(z)$  less likely to have an inside root when  $\frac{1-g_\eta}{1-g_\varepsilon}$  is relatively large. This observation suggests that intuitively, a higher  $\tau_\varepsilon$  has the following two opposing effects:

1. The extra precision leads to a better estimate of the fundamental and a stronger response to the fundamental, a higher  $g_\eta$ . This tends to make the equilibrium invertible.
2. The response to the common noise,  $g_\varepsilon$ , also increase since agents rely relatively more on the public signal  $z_t$ . This makes the information content of the aggregate action  $a_{t-1}$  closer to that of the public signal  $z_t$ . Put differently, there is less differential information contained in the aggregate outcome in comparison with the public signal, and this tends to make the equilibrium non-invertible.

To further appreciate this last point, consider the special case in which  $d(L) = 1/(1 - \rho L)$  and  $\tau_\omega \rightarrow \infty$ . The information structure, then, reduces to the one from Section 3 with the addition of a public signal. In this case, the invertibility condition (5.4) becomes

$$\alpha < 1 - \frac{\rho}{\tau_u},$$

which is identical to condition (3.3). This may seem puzzling at first, as the precision of the public signal plays no role in determining invertibility. However, this is because the two forces discussed above exactly cancel each other. In the extreme, when  $\tau_\varepsilon$  goes to infinity, agents can infer the fundamental almost perfectly using the public signal. On the other hand, they correspondingly discard their private signals, and the aggregate outcome contains no more information than the one already obtained with the public signal. These effects cancel, leaving open the possibility that the equilibrium is not invertible.

In contrast, when  $\tau_\omega$  is finite, agents always use their private signals to learn about their idiosyncratic fundamental. It follows that the aggregate outcome necessarily aggregates the information contained in private signals, which differentiates itself from the public signal  $z_t$ . The two forces do not cancel each other, and the precision of the public signal,  $\tau_\varepsilon$ , does matter for the determination of invertibility.

## 5.2 Forward Complementarities

The best-response function in equation (5.1) only allows for static strategic complementarities, that is, agent  $i$ 's action depends on the current aggregate action. Here, we extend the analysis to allow for

arbitrary forward-looking complementarities, that is, agent  $i$ 's action can depend on future aggregate actions or on their own future actions in a flexible way. We consider the following best-response function,

$$a_{it} = \varphi \mathbb{E}_{it}[\xi_{it}] + \alpha \mathbb{E}_{it}[a_t] + \mathbb{E}_{it}[\gamma(L)a_t] + \mathbb{E}_{it}[\beta(L)a_{it}], \quad (5.7)$$

where

$$\gamma(L) \equiv \sum_{k=1}^{\infty} \gamma_k L^{-k}, \quad \beta(L) \equiv \sum_{k=1}^{\infty} \beta_k L^{-k}, \quad \text{and} \quad |\alpha| + \|\gamma(L)\| + \|\beta(L)\| < 1.$$

and we impose a relatively weak condition on the parameters that guarantees existence of the equilibrium.<sup>22</sup>

Even though the model structure is more sophisticated, it turns out that when the equilibrium is invertible, the general best-response function in condition (5.7) collapses to the static best response in condition (5.1) with a modified fundamental, as forward-looking higher-order expectations collapse to first-order expectations in this scenario. The following proposition formalizes the required transformation.

**Proposition 5.2.** *If the equilibrium is invertible, then the actions under best response (5.7) are observationally equivalent to those under the following transformed best response*

$$a_{it} = \varphi \mathbb{E}_{it}[\tilde{\xi}_{it}] + \alpha \mathbb{E}_{it}[a_t], \quad \text{with} \quad \tilde{\xi}_{it} \equiv \frac{1 - \alpha}{1 - \alpha - \gamma(L) - \beta(L)} \xi_{it}.$$

*Proof.* See Appendix D. □

With this transformation, Theorem 1 can be applied to the general class of best-response functions described by condition (5.7). The fact that there are forward complementarities does not change the fact that there always exists a threshold level for the degree of *static* strategic complementarity,  $\bar{\alpha}$ , such that, if  $\alpha \geq \bar{\alpha}$ , the equilibrium is not invertible.

The effects of forward complementarities on the invertibility of the equilibrium, however, are not as simple as in the static case. In Appendix G we provide an analysis of the invertibility condition with an ARMA (1,1) fundamental process, which is fairly complicated as it hinges on the interaction between the fundamental and the coordination structure. In Appendix H we show how Theorem 1 can be further extended to economies with both forward and backward complementarities which encompasses many environments considered in the DSGE literature.

## 6. APPLICATIONS IN NEW KEYNESIAN MODELS

In this section, we apply our results to a New Keynesian model with incomplete information. We start with the analysis that treats the demand block and the supply block independently, and focus

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<sup>22</sup>In these conditions,  $\|\cdot\|$  denotes the operator norm.



on how the key parameters such as the price stickiness and the MPC that governs the GE feedback effects also modify how much information is aggregated. We then unify the two blocks to explore the implications of a more dovish Taylor rule on the aggregate outcomes.

## 6.1 Price stickiness and MPC

Consider the following univariate best response

$$a_{it} = \varphi \mathbb{E}_{it}[\xi_t] + \alpha \mathbb{E}_{it}[a_t] + \beta \mathbb{E}_{it}[a_{it}].$$

This setting can nest the two building blocks of the New Keynesian model, the Dynamic IS curve, and the New Keynesian Philips curve (NKPC). Relaxing the common-knowledge foundations of the New Keynesian model along the lines of [Angeletos and Lian \(2018\)](#) yields<sup>23</sup>

$$\pi_{it} = \kappa \mathbb{E}_{it}[mc_t] + (1 - \theta) \mathbb{E}_{it}[\pi_t] + \delta \theta \mathbb{E}_{it}[\pi_{it+1}], \quad (6.1)$$

$$c_{it} = -\zeta(1 - \text{mpc}) \mathbb{E}_{it}[r_t] + \text{mpc} \mathbb{E}_{it}[c_t] + (1 - \text{mpc}) \mathbb{E}_{it}[c_{i,t+1}]. \quad (6.2)$$

In the NKPC,  $mc_t$  is the real marginal cost,  $\theta$  is the Calvo parameter, and  $\delta$  is the discount rate. In the dynamic IS curve,  $r_t$  is the real rate,  $\zeta$  is the intertemporal elasticity of substitution,  $\text{mpc}$  is the marginal propensity to consume. With complete information, these two conditions reduce to their familiar textbook versions. Crucially, the price stickiness  $\theta$  and the MPC control the degree of static strategic complementarity in firms' pricing decision and consumers' saving-consumption decision.

Here, we treat the marginal cost,  $mc_t$ , and the real rate,  $r_t$ , as exogenous fundamentals, and therefore the supply block (6.1) and the demand block (6.2) do not interact with each other and can be treated separately, as in [Nimark \(2008\)](#) and [Angeletos and Huo \(2018\)](#). As in our baseline specification, we assume that the fundamentals  $mc_t$  and  $r_t$  follow AR(1) processes. Every period, firms and consumers observe an exogenous signal about the fundamental with private noises, and the past realization of  $\pi_t$  and  $c_t$ , respectively.

Applying Proposition 5.2 and Theorem 1 leads to the following observation.

**Corollary 2.** *For a fixed level of private noise, inflation is noninvertible when prices are sufficiently flexible (lower  $\theta$ ), and aggregate consumption is noninvertible when MPC is sufficiently high (higher  $\text{mpc}$ ).*

*Proof.* See Appendix E. □

The underlying logic for this result can be understood as follows: a lower  $\theta$  implies more frequent price adjustments. As a result, for the firms that can adjust their prices, there is a greater need to worry about other firms' price-setting decisions. A higher MPC makes consumption more sensitive

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<sup>23</sup>Appendix E presents a derivation of these equations together with the proof of Corollary 2.

to income changes, which leads to a stronger dependence on aggregate demand. Both of these effects map to a higher degree of static strategic complementarity when the best responses are transformed according to Proposition 5.2.

## 6.2 Role of aggressiveness of nominal interest rate response

In the previous example, we treat inflation and output separately. In this section, we show that the main insights extend to the case where the supply and demand block of the economy interact with each other. Particularly, we show that when the central bank follows a standard Taylor rule, a less aggressive response of the nominal interest rate to inflation tends to reduce the amount of information aggregated by output and inflation, yielding a non-invertible equilibrium.

We extend the model in the following way,

$$\pi_{it} = \kappa \mathbb{E}_{it}[c_t + \xi_t^s] + (1 - \theta) \mathbb{E}_{it}[\pi_t] + \delta \theta \mathbb{E}_{it}[\pi_{t+1}], \quad (6.3)$$

$$c_{it} = -\zeta(1 - \text{mpc}) \mathbb{E}_{it}[i_t - \pi_{t+1} + \xi_t^d] + \text{mpc} \mathbb{E}_{it}[c_t] + (1 - \text{mpc}) \mathbb{E}_{it}[c_{i,t+1}], \quad (6.4)$$

$$i_t = \phi_\pi \pi_t. \quad (6.5)$$

The first equation is the NKPC, modified to allow a real marginal cost proportional to aggregate output, and subject to a cost-push shock  $\xi_t^s$ . The second equation is the dynamic IS curve replacing real interest rate by the difference of nominal interest rates and expected future inflation, and augmented with a preference shock  $\xi_t^d$ . The third equation is the Taylor rule for the nominal interest rate, where  $\phi_\pi$  controls how aggressive monetary policy is against inflation.

In terms of information, we allow individual firms and consumers to observe private signals about the demand and supply shocks,  $x_{i,t}^s = \xi_t^s + u_{i,t}^s$ ,  $x_{i,t}^d = \xi_t^d + u_{i,t}^d$ , as well as past inflation and output realizations. To simplify the analysis, we assume that both  $\xi_t^s$  and  $\xi_t^d$  follows an AR(1) process with persistence  $\rho$ , and that  $u_{i,t}^s$  and  $u_{i,t}^d$  are both i.i.d shocks with variance  $\tau^{-1}$ .<sup>24</sup>

System (6.3)-(6.5) effectively consists of a forward-looking network game. As a result, the strength of GE effects, or the degree strategic complementarity, no longer depends on a single scalar but on all the relevant structural parameters that control within-group and cross-group dependences. Here, we highlight the role of  $\phi_\pi$ . When the nominal interest rate is passive (such as in the zero-lower bound), higher inflation induces higher demand, which further pushes up marginal costs and inflation. A higher  $\phi_\pi$  suppresses aggregate demand's response and mutes this reinforcing mechanism, which implies a higher degree of strategic *substitutability*.

With perfect information, varying  $\phi_\pi$  only matters for the magnitude of the responses of output and inflation to shocks. With incomplete information and observation about endogenous outcomes, varying  $\phi_\pi$  also changes the information content of  $c_t$  and  $\pi_t$ . As a higher  $\phi_\pi$  implies higher strategic

<sup>24</sup>The main conclusion of our results on the comparative statics on  $\phi_\pi$  does not depend on the symmetry assumption.

substitutability, it makes agents rely more on their private signals. Thus, the economy is more likely to be invertible in the sense that, by observing  $\{c^t, \pi^t\}$ , one can perfectly infer the demand and supply shocks, i.e.,<sup>25</sup>

$$\mathbb{E}[\xi_t^s | c^t, \pi^t] = \xi_t^s, \quad \text{and} \quad \mathbb{E}[\xi_t^d | c^t, \pi^t] = \xi_t^d.$$

This argument is formalized in the following proposition.

**Proposition 6.1.** *The outcomes  $\{\pi^t, c^t\}$  perfectly aggregate information only if  $\phi_\pi$  exceeds the threshold  $\Phi$  given by*

$$\Phi = \begin{cases} \frac{\rho^2 - \theta(1 - mpc)\tau^2}{\kappa\zeta(1 - mpc)\tau^2}, & \text{when } (1 - \theta - mpc)^2 < 4\kappa\zeta(1 - mpc)\phi_\pi, \\ \frac{\tau\rho(1 + \theta - mpc) - \rho^2 - \theta(1 - mpc)\tau^2}{\kappa\zeta(1 - mpc)\tau^2}, & \text{otherwise.} \end{cases} \quad (6.6)$$

*Proof.* See Appendix F. □

The left panel of Figure 7 shows how equilibrium invertibility depends on the aggressiveness of the nominal interest rate response,  $\phi_\pi$ , and the precision of the exogenous signals received by the agents,  $\tau$ . Following Proposition 6.1, a higher  $\phi_\pi$  makes endogenous outcomes more likely to be invertible. The right panel presents an example of an impulse response of inflation to the supply shock  $\xi_t^s$  in the non-invertible region. In the invertible region, the IRF of inflation differs from its perfect information counterpart only in the initial period, that is,

$$\pi_t = \pi_t^* - \varphi\eta_t^s,$$

where  $\pi_t^*$  is the perfect information outcome, and  $\varphi$  is a constant modifying the response on impact.<sup>26</sup>

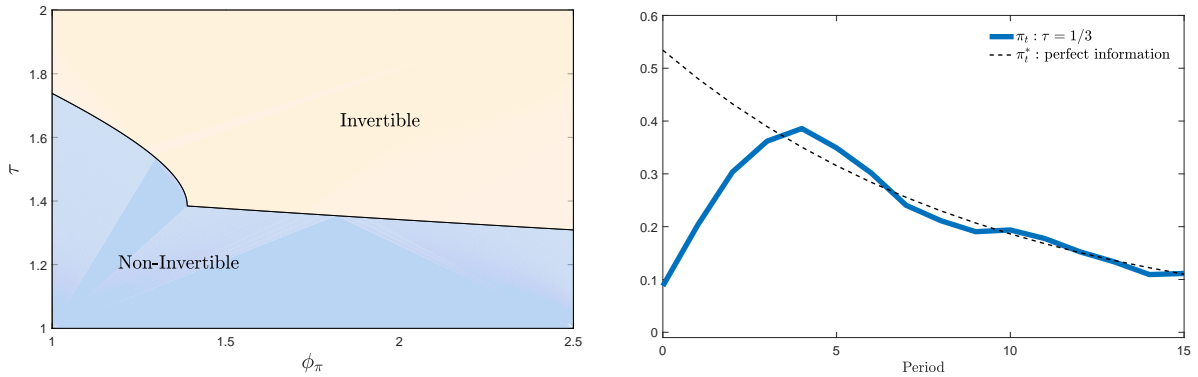


Figure 7: Invertibility region and Example of a non-invertible impulse response function

Parameters:  $\kappa = 0.05$ ,  $\theta = 0.4$ ,  $\delta = 0.99$ ,  $\zeta = 1$ ,  $mpc = 0.1$ , and  $\rho = 0.9$ . For the right panel,  $\phi_\pi = 1$  is used to generate the IRF.

<sup>25</sup>Technically, invertibility requires the equilibrium process  $\begin{bmatrix} c_t \\ \pi_t \end{bmatrix} = \mathbf{g}(L) \begin{bmatrix} \eta_t^s \\ \eta_t^d \end{bmatrix}$  to satisfy the condition that  $\det[\mathbf{g}(L)]$  does not contain any inside root.

<sup>26</sup>The formulas for  $\pi_t^*$  and  $\varphi$  can be found in Appendix F.

In contrast, in the non-invertible region, inflation displays the now familiar early underreaction and the following oscillatory pattern. This suggests that the central bank's policy rule matters not only for the magnitude of aggregate outcome's response to shocks but also disciplines their entire dynamic patterns. Interestingly, according to our theory, a more hawkish monetary policy, such as that followed in the post-Volker era, is predicted to eliminate the additional persistence of inflation due to imperfect information aggregation, which is consistent with the empirical findings in [Fuhrer \(2010\)](#) and [Cogley, Primiceri, and Sargent \(2010\)](#).

## 7. CONCLUSION

We show that, with dispersed information, even when past aggregate outcomes are perfectly observed, the underlying states may not be fully revealed. The extent to which endogenous outcomes help aggregate information is increasing in the precision of private signals about economic fundamentals and is decreasing in the strength of general equilibrium feedback effects. With imperfect information aggregation, the equilibrium dynamics oscillate around the economic fundamental, and the forecast errors display an initial underreaction and a delayed overreaction which is consistent with the empirical evidence in [Angeletos, Huo, and Sastry \(2020\)](#). Together with the solution to the exogenous information environment, our results provide a second benchmark to characterize properties of equilibrium under incomplete information.

When the theory is applied to familiar macro models, it leads to several applied implications. In monetary models à la [Woodford \(2003\)](#), the GE feedback effect comes from firms' pricing complementarity. When it is strong enough, monetary shocks can have persistent effects even when firms observe past prices. In standard NKPC models, a less aggressive monetary policy induces a stronger GE feedback effect between inflation and output, and can result in hump-shaped and oscillatory responses of aggregate variables.

Left outside this paper is the extension beyond the standard linear-Gaussian framework. In an asset pricing setting, [Albagli, Hellwig, and Tsyvinski \(2015\)](#) allow a much more flexible payoff structure and shock distribution. [Hassan and Mertens \(2015\)](#) also incorporate dispersed information into a non-linear DSGE model, and quantify the informational role of stock prices. These works allow agents to observe past shocks, and therefore the learning is essentially static. It would be interesting to explore the extent to which agents can learn from the past in a non-linear environment.

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## Appendix

### A. PROOF OF PROPOSITION 3.1

Suppose that the stochastic process for  $p_t$  is invertible, then observing  $\{p_k\}_{k=-\infty}^{t-1}$  perfectly reveals the underlying aggregate shocks  $\{\eta_k\}_{k=-\infty}^{t-1}$  and, therefore,  $\{q_k\}_{k=-\infty}^{t-1}$ , so that the only shock the firms are uncertain about is the current  $\eta_t$ . Guess that the equilibrium policy function has state variables  $x_{it}$  and  $q_{t-1}$ , that is guess that firm  $i$ 's policy function can be written as

$$p_{it} = \phi_x x_{it} + \phi_q q_{t-1},$$

for some scalars  $\phi_x$  and  $\phi_q$ . It follows that, in aggregate terms,

$$p_t = \phi_x q_t + \phi_q q_{t-1}.$$

To verify the guess, notice that, since  $x_{it} - \rho q_{t-1}$  is a noisy signal about  $\eta_t$  with precision  $\tau$ , we have that

$$\mathbb{E}_{it}[q_t] = \frac{\rho}{1+\tau} q_{t-1} + \frac{\tau}{1+\tau} x_{it}.$$

Substituting these results into the best-response function (2.2) we obtain

$$p_{it} = \frac{\tau((1-\alpha) + \alpha\phi_x)}{1+\tau} x_{it} + \left( \frac{\rho((1-\alpha) + \alpha\phi_x)}{1+\tau} + \alpha\phi_q \right) q_{t-1},$$

which implies the following consistency requirement

$$\phi_q = \frac{\rho}{1+(1-\alpha)\tau}, \quad \text{and} \quad \phi_x = \frac{(1-\alpha)\tau}{1+(1-\alpha)\tau}.$$

Hence, for  $p_t$  to indeed follow an invertible process it is necessary and sufficient that  $g(L) = \frac{\phi_x + \phi_q L}{1 - \rho L}$  not have an inside root, or that  $|\phi_q/\phi_x| < 1$  which implies the result.

### B. PROOF OF PROPOSITION 3.2

The first claim follows directly from the proof of Proposition 3.1. To establish the second claim we follow a significantly more involved argument. To facilitate reading it we include the proof of the necessary lemmas at the end of this section.

For a contradiction, suppose there is a finite-state representation, then the law of motion of the aggregate action can be written as

$$p_t = g(L)\eta_t = C(L)h(L)\eta_t,$$

where  $h(L)$  is analytic and does not contain any inside root, and  $C(z)$  is given by

$$C(z) = \prod_{i=1}^n (z - \lambda_i).$$

where  $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$  are inside roots of  $g(z)$ . The signal structure can be written as follows

$$\begin{bmatrix} x_{it} \\ p_{t-1} \end{bmatrix} = \mathbf{\Gamma}(L) \begin{bmatrix} u_{it} \\ \eta_t \end{bmatrix} \equiv \begin{bmatrix} \tau^{-1/2} & \frac{1}{1-\rho L} \\ 0 & Lg(L) \end{bmatrix} \begin{bmatrix} u_{it} \\ \eta_t \end{bmatrix}.$$

The determinant of  $\mathbf{\Gamma}(L)$  is

$$\det[\mathbf{\Gamma}(L)] = \tau^{-1/2} L C(L) h(L),$$

and it contains inside roots  $\{\lambda_1, \dots, \lambda_n, \lambda_{n+1}\}$ , with  $\lambda_{n+1} \equiv 0$ . Denote the Blaschke matrix by

$$\mathbf{B}(L; \lambda) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1-\lambda L}{L-\lambda} \end{bmatrix},$$

and let the fundamental representation of the signal process be given by

$$\mathbf{\Gamma}^*(L) \varepsilon_{it} = \mathbf{\Gamma}(L) \begin{bmatrix} u_{it} \\ \eta_t \end{bmatrix},$$

where

$$\varepsilon_{it} \equiv \mathbf{A}(L) \begin{bmatrix} u_{it} \\ \eta_t \end{bmatrix}, \quad \mathbf{A}(L) \equiv \mathbf{B}'(L^{-1}; \lambda_{n+1}) \mathbf{W}'_{\lambda_{n+1}} \cdots \mathbf{B}'(L^{-1}; \lambda_1) \mathbf{W}'_{\lambda_1}, \quad \mathbf{\Gamma}^*(L) \equiv \mathbf{\Gamma}(L) \mathbf{A}'(L^{-1}).$$

and  $\{\mathbf{W}_{\lambda_i}\}$  are the rotation matrices that satisfy  $\mathbf{W}_{\lambda_i} \mathbf{W}'_{\lambda_i} = \mathbf{I}$ . Next, define the following matrices recursively

$$\begin{aligned} \mathbf{\Gamma}_0(L) &\equiv \mathbf{\Gamma}(L), \\ \mathbf{\Gamma}_k(L) &\equiv \mathbf{\Gamma}_{k-1}(L) \mathbf{W}_{\lambda_k} \mathbf{B}(L; \lambda_k). \end{aligned}$$

The following lemma characterizes useful properties of  $\mathbf{\Gamma}_k(L)$ .

**Lemma 4.** *The matrix  $\mathbf{\Gamma}_k(L)$  is given by*

$$\mathbf{\Gamma}_k(L) = \begin{bmatrix} \gamma_1^k(L) & \gamma_2^k(L) \\ \gamma_3^k(L) g(L) & \gamma_4^k(L) g(L) \end{bmatrix}$$

with all  $\gamma_i^k(L)$  independent of  $g(L)$ . Moreover,

$$\delta_k(z) \equiv \frac{\gamma_1^k(z)}{\gamma_2^k(z)}$$

satisfies the following recursive structure:

$$\delta_0(z) = \tau^{-1/2}(1 - \rho z), \tag{B.1}$$

$$\delta_k(z) = \frac{1 + \delta_{k-1}(\lambda_k) \delta_{k-1}(z)}{\delta_{k-1}(\lambda_k) - \delta_{k-1}(z)} \frac{z - \lambda_k}{1 - \lambda_k z}. \tag{B.2}$$

Finally, for  $k \geq 2$ , there exists some constant  $d_k$  such that

$$\delta_{k-1}(z) = \delta_{k-1}(\lambda_k) + d_k \frac{z - \lambda_k}{1 - \lambda_{k-1} z}. \tag{B.3}$$

Using the recursive structure of  $\delta_k(z)$ , it is straightforward to verify that  $\mathbf{A}(z)$  can be written as

$$\mathbf{A}(z) = c_1 \mathbf{\Phi}(z) = \mathbf{A}_{n+1}(z) \mathbf{A}_n(z) \dots \mathbf{A}_2(z) \mathbf{A}_1(z),$$

with

$$c_1 = \prod_{k=1}^{n+1} \frac{\gamma_2^{k-1}(\lambda_k)}{\sqrt{\gamma_1^{k-1}(\lambda_k)^2 + \gamma_2^{k-1}(\lambda_k)^2}},$$

$$\mathbf{A}_k(z) = \frac{\sqrt{\gamma_1^{k-1}(\lambda_k)^2 + \gamma_2^{k-1}(\lambda_k)^2}}{\gamma_2^{k-1}(\lambda_k)} \mathbf{B}'(z^{-1}; \lambda_k) \mathbf{W}'_{\lambda_k} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{z-\lambda_k}{1-\lambda_k z} \end{bmatrix} \begin{bmatrix} \delta_{k-1}(\lambda_k) & 1 \\ -1 & \delta_{k-1}(\lambda_k) \end{bmatrix}.$$

The following lemma characterizes a useful property of  $\mathbf{\Phi}(z)$ .

**Lemma 5.** *The elements of  $\mathbf{\Phi}(z)$  satisfy*

$$\Phi_{12}(z) = -\frac{\prod_{k=1}^{n+1} (\delta_{k-1}(\lambda_k)^2 + 1)}{c_2} \frac{\delta_0(z^{-1})}{H(z^{-1})} + \Phi_{22}(z) \delta_{n+1}(z^{-1}),$$

$$\Phi_{22}(z) = c_2 \frac{H(z) \delta_0(z^{-1}) \delta_{n+1}(z) + 1}{\delta_0(z) \delta_0(z^{-1}) + 1},$$

where  $H(z) \equiv \prod_{k=1}^{n+1} \frac{z-\lambda_k}{1-\lambda_k z}$ , and  $c_2$  is some constant.

The equivalence result in [Huo and Pedroni \(2020\)](#) implies that

$$p_{it} = \tilde{\mathbb{E}}_{it}[q_t],$$

where  $\tilde{\mathbb{E}}_{it}$  is the expectation conditional on the same information set but with a precision of private signals misperceived to be  $\tau \equiv (1 - \alpha)\tau$ . Then, the aggregate action is

$$p_t = \int \tilde{\mathbb{E}}_{it}[q_t] di.$$

Since the Hansen-Sargent formula implies

$$\tilde{\mathbb{E}}_{it}[q_t] = \frac{1}{\rho} \left( \frac{\mathbf{\Gamma}(L)}{L} - \frac{\mathbf{\Gamma}^*(0) \mathbf{A}(L)}{L} \right)_{1st \text{ row}} \begin{bmatrix} u_{it} \\ \eta_t \end{bmatrix},$$

we obtain the following fixed point problem,

$$g(z) = \frac{1}{\rho} \left[ \frac{\mathbf{\Gamma}(z)}{z} - \frac{\mathbf{\Gamma}^*(0) \mathbf{A}(z)}{z} \right]_{12} = \frac{1}{\rho z} \left( \frac{1}{1 - \rho z} - S(z) \right),$$

or

$$h(z) = \frac{1}{\rho} \frac{1}{zC(z)} \left( \frac{1}{(1 - \rho z)} - S(z) \right),$$

where

$$S(z) \equiv \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{\Gamma}^*(0) \mathbf{A}(z) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1 \left( \gamma_1^{n+1}(0) \Phi_{12}(z) + \gamma_2^{n+1}(0) \Phi_{22}(z) \right).$$

For  $h(z)$  to be an equilibrium, it has to be that

$$\frac{1}{(1 - \rho\lambda_i)} = S(\lambda_i), \quad \text{for all } i \in \{1, \dots, n\}, \quad (\text{B.4})$$

so that the poles in  $C(z)$  can be removed.

Next, we show that with a finite number of inside roots, there does not exist an equilibrium.

**Case 1:**  $n = 1$ .. First let  $x \equiv \rho + \frac{1+\tau}{\rho}$  and notice that  $|\rho| < 1$  and  $\tau > 0$  imply  $|x| > 2$ . In this case  $S(z)$  takes a simple form and we can calculate

$$S(\lambda_1) - \frac{1}{1 - \rho\lambda_1} = \frac{\tau\lambda_1}{(1 - \lambda_1\rho)} \frac{x - \lambda_1}{(1 - \lambda_1^2) + (x - 2\lambda_1)(\rho - x)},$$

so that equation (B.4) implies that  $\lambda_1 = x$  which is outside the unit circle.

**Case 2:**  $n = 2$ .. Suppose that  $\lambda_1 \neq \lambda_2$ , then

$$\frac{S(\lambda_1)}{S(\lambda_2)} - \frac{1 - \rho\lambda_2}{1 - \rho\lambda_1} = \frac{\tau(\lambda_2 - \lambda_1)}{(1 - \rho\lambda_1)} \frac{(x - \lambda_1)(x - \lambda_2) - 1}{(1 - \lambda_1^2)(\rho - x + \lambda_2) - (x - 2\lambda_1)((x - \lambda_2)(\rho - x) + 1)},$$

and equation (B.4) for  $i = \{1, 2\}$  implies that  $(x - \lambda_1)(x - \lambda_2) = 1$ , which implies that either  $|\lambda_1| > 1$  or  $|\lambda_2| > 1$ . Next, if  $\lambda_1 = \lambda_2 = \lambda$ , we have that

$$S(\lambda) - \frac{1}{1 - \rho\lambda} = \frac{\tau\lambda}{1 - \lambda\rho} \frac{-3\lambda^2 + 3x\lambda - (x^2 - 1)}{(1 - \lambda^2)(4\lambda - x(1 + \lambda^2)) + (4x\lambda - x^2 + \lambda^4 - 6\lambda^2 + 1)(\rho - x)},$$

and equation (B.4) implies that  $3\lambda^2 - 3x\lambda + x^2 = 1$ . Notice that the discriminant of this quadratic equation on  $\lambda$  is  $9x^2 - 12(x^2 - 1)$  and that it is negative whenever  $|x| > 2$ . Therefore, the solutions are complex. Complex  $\lambda$ 's are allowed but necessitate a conjugate which is not possible in this case since we have assumed  $\lambda_1 = \lambda_2$ .

**Case 3:**  $n > 2$ .. From the definition of  $S(z)$  and Lemma 5, it follows that

$$S(\lambda_i) = c_1 c_2 \gamma_2^{n+1}(0) \frac{1 + \delta_{n+1}(0)\delta_{n+1}(\lambda_i^{-1})}{1 + \delta_0(\lambda_i)\delta_0(\lambda_i^{-1})}.$$

Equation (B.3) together with the fact that  $\lambda_{n+1} = 0$  implies that

$$\delta_{n+1}(z) = \delta_{n+1}(0) + d_{n+1}z,$$

for some constant  $d_{n+1}$ . Thus, we can rewrite  $S(\lambda_i)$  as

$$S(\lambda_i) = c_1 c_2 \gamma_2^{n+1}(0) \frac{1 + \delta_{n+1}(0)(\delta_{n+1}(0) + d_{n+1}\lambda_i^{-1})}{1 + \delta_0(\lambda_i)\delta_0(\lambda_i^{-1})}.$$

Suppose that the solution to the system of equations (B.4) includes  $\lambda_i, \lambda_j, \lambda_k$  different from each other and all inside the unit circle. It follows that

$$\frac{S(\lambda_i)}{S(\lambda_j)} = \frac{1 - \rho\lambda_j}{1 - \rho\lambda_i}, \quad \text{or} \quad \frac{1 + \delta_{n+1}(0)(\delta_{n+1}(0) + d_{n+1}\lambda_i^{-1})}{1 + \delta_0(\lambda_i)\delta_0(\lambda_i^{-1})} = \frac{1 + \delta_{n+1}(0)(\delta_{n+1}(0) + d_{n+1}\lambda_j^{-1})}{1 + \delta_0(\lambda_j)\delta_0(\lambda_j^{-1})},$$

which can be written as

$$\delta_{n+1}(0) (\tau + (1 - \rho\lambda_i) (1 - \rho\lambda_j)) + ((1 - \rho\lambda_i) (1 - \rho\lambda_j) + \lambda_i\lambda_j\tau) \rho (1 + \delta_{n+1}^2(0)) = 0.$$

Suppose that  $\tau + (1 - \rho\lambda_i) (1 - \rho\lambda_j) \neq 0$ , then

$$\frac{(1 - \lambda_i\lambda_j) \tau}{\tau + (1 - \rho\lambda_i) (1 - \rho\lambda_j)} = 1 - \frac{d_{n+1}\delta_{n+1}(0)}{\rho (1 + \delta_{n+1}^2(0))}.$$

Similarly, if  $\tau + (1 - \rho\lambda_i) (1 - \rho\lambda_k) \neq 0$ , then

$$\frac{(1 - \lambda_i\lambda_k) \tau}{\tau + (1 - \rho\lambda_i) (1 - \rho\lambda_k)} = 1 - \frac{d_{n+1}\delta_{n+1}(0)}{\rho (1 + \delta_{n+1}^2(0))}.$$

Combining the two conditions above, we have

$$\frac{(1 - \lambda_i\lambda_j)}{\tau + (1 - \rho\lambda_i) (1 - \rho\lambda_j)} = \frac{(1 - \lambda_i\lambda_k)}{\tau + (1 - \rho\lambda_i) (1 - \rho\lambda_k)},$$

which implies

$$\tau\lambda_i + (1 - \lambda_i\rho)(1\lambda_i - \rho) = 0 \quad \Rightarrow \quad 1 + \delta_0(\lambda_i)\delta_0(\lambda_i^{-1}) = 0.$$

This cannot be the case since  $S(\lambda_i)$  must be a finite number. Therefore, if there exists a solution, it has to be that

$$\tau + (1 - \rho\lambda_i) (1 - \rho\lambda_j) = 0.$$

This condition, however, implies that  $\lambda_i$  and  $\lambda_j$  cannot be simultaneously within the unit circle.

Next, suppose there exists a solution with  $\lambda_i = \lambda$ , for all  $i \in \{1, \dots, n\}$  with  $|\lambda| < 1$ . Let  $\{\lambda_{1,k}\}_{k=0}^{\infty}, \dots, \{\lambda_{n,k}\}_{k=0}^{\infty}$  be  $n$  sequences such that  $\lim_{k \rightarrow \infty} \lambda_{i,k} = \lambda$  and  $\lambda_{i,k} \neq \lambda_{j,k}$  for all  $i, j \in \{1, \dots, n\}$  and all  $k \geq 0$ . Define

$$\omega_k \equiv \sum_{i=1}^n \left| S(\lambda_{i,k}) - \frac{1}{1 - \rho\lambda_{i,k}} \right|.$$

By continuity of  $S(\cdot)$ ,  $\omega_k$  approaches 0 as  $k$  goes to infinity, since  $\lambda_{i,k}$  approaches to  $\lambda$  for all  $i \in \{1, \dots, n\}$ . However, since  $\lambda_{i,k} \neq \lambda_{j,k}$ , as established above, only if some  $\lambda_{i,k}$  are outside the unit circle, can  $\omega_k$  approach 0. Since  $|\lambda| < 1$ , as  $k$  goes to infinity, all  $|\lambda_{i,k}| < \delta$  for any  $\delta < 1$ , which implies that  $\omega_k$  cannot be close to zero. This is a contradiction. The case where all  $\lambda_i$  equal to each other, except for one, can be dealt with in a similar way, which concludes the proof.

## B.1 Proof of Lemma 4

We prove this lemma by induction. For  $k = 0$ , this is clearly the case. For  $k \geq 1$ , suppose that

$$\Gamma_k(z) = \begin{bmatrix} \gamma_1^k(z) & \gamma_2^k(z) \\ \gamma_3^k(z) g(z) & \gamma_4^k(z) g(z) \end{bmatrix}.$$

Then, it follows that

$$\Gamma_{k+1}(z) = \Gamma_{k-1}(z) \mathbf{W}_{\lambda_{k+1}} \mathbf{B}(z; \lambda_{k+1}),$$

with

$$\mathbf{W}_{\lambda_{k+1}} = \frac{1}{\sqrt{\gamma_1^k(\lambda_{k+1})^2 + \gamma_2^k(\lambda_{k+1})^2}} \begin{bmatrix} \gamma_1^k(\lambda_{k+1}) & -\gamma_2^k(\lambda_{k+1}) \\ \gamma_2^k(\lambda_{k+1}) & \gamma_1^k(\lambda_{k+1}) \end{bmatrix}.$$

So that

$$\begin{aligned} \Gamma_{k+1}(z) &= \frac{1}{\sqrt{\gamma_1^k(\lambda_{k+1})^2 + \gamma_2^k(\lambda_{k+1})^2}} \begin{bmatrix} \gamma_1^k(z) & \gamma_2^k(z) \\ \gamma_3^k(z) g(z) & \gamma_4^k(z) g(z) \end{bmatrix} \begin{bmatrix} \gamma_1^k(\lambda_{k+1}) & -\gamma_2^k(\lambda_{k+1}) \\ \gamma_2^k(\lambda_{k+1}) & \gamma_1^k(\lambda_{k+1}) \end{bmatrix} \\ &= \frac{\begin{bmatrix} (\gamma_1^k(\lambda_{k+1}) \gamma_1^k(z) + \gamma_2^k(\lambda_{k+1}) \gamma_2^k(z)) & (\gamma_1^k(\lambda_{k+1}) \gamma_2^k(z) - \gamma_2^k(\lambda_{k+1}) \gamma_1^k(z)) \frac{1-z\lambda_{k+1}}{z-\lambda_{k+1}} \\ (\gamma_1^k(\lambda_{k+1}) \gamma_2^k(z) + \gamma_2^k(\lambda_{k+1}) \gamma_4^k(z)) g(z) & (\gamma_1^k(\lambda_{k+1}) \gamma_4^k(z) - \gamma_2^k(\lambda_{k+1}) \gamma_3^k(z)) \frac{1-z\lambda_{k+1}}{z-\lambda_{k+1}} g(z) \end{bmatrix}}{\sqrt{\gamma_1^k(\lambda_{k+1})^2 + \gamma_2^k(\lambda_{k+1})^2}} \end{aligned}$$

which has the desired structure. Note, moreover, that  $\gamma_1^{k+1}$  and  $\gamma_2^{k+1}$  satisfy

$$\begin{aligned} \gamma_1^{k+1}(z) &= \frac{\gamma_1^k(\lambda_{k+1}) \gamma_1^k(z) + \gamma_2^k(\lambda_{k+1}) \gamma_2^k(z)}{\sqrt{\gamma_1^k(\lambda_{k+1})^2 + \gamma_2^k(\lambda_{k+1})^2}}, \\ \gamma_2^{k+1}(z) &= \frac{\gamma_1^k(\lambda_{k+1}) \gamma_2^k(z) - \gamma_2^k(\lambda_{k+1}) \gamma_1^k(z)}{\sqrt{\gamma_1^k(\lambda_{k+1})^2 + \gamma_2^k(\lambda_{k+1})^2}} \frac{1-z\lambda_{k+1}}{z-\lambda_{k+1}}. \end{aligned}$$

It follows from these recursions that  $\delta_k(z)$  satisfies equation (B.2). To prove equation (B.3), first notice that, for  $k = 1$ , it follows from (B.1) and (B.2) that

$$\delta_1(z) = \delta_1(\lambda_1) + \frac{(1-\rho\lambda_1)(\lambda_1-\rho) + \lambda_1\tau}{\sqrt{\tau}\rho(1-\lambda_1^2)} \frac{z-\lambda_1}{1-\lambda_1z}.$$

Next, suppose that there exists  $d_k$  such that

$$\delta_k(z) = \delta_k(\lambda_{k+1}) + d_k \frac{z-\lambda_{k+1}}{1-\lambda_{k+1}z},$$

then, equation (B.2) implies that

$$\delta_{k+1}(z) = \delta_{k+1}(\lambda_{k+1}) + \frac{(1 + \delta_k(\lambda_{k+1})^2)(\lambda_k - \lambda_{k+1}) - d_k(1 - \lambda_{k+1}^2) \delta_k(\lambda_{k+1})}{d_k(1 - \lambda_{k+1}^2)} \frac{z - \lambda_{k+1}}{1 - \lambda_{k+1}z},$$

which, again by induction, establishes the result.  $\square$

## B.2 Proof of Lemma 5

From the definition of  $\mathbf{A}_k(z)$  and equation (B.2), it follows that

$$\mathbf{A}_k(z) \begin{bmatrix} 1 \\ -\delta_{k-1}(z) \end{bmatrix} = (\delta_{k-1}(\lambda_k) - \delta_{k-1}(z)) \begin{bmatrix} 1 \\ -\delta_k(z) \end{bmatrix}.$$

Define

$$H(z) \equiv \prod_{k=1}^{n+1} \frac{z - \lambda_k}{1 - \lambda_k z}, \quad \text{and} \quad G(z) \equiv \prod_{k=1}^{n+1} (\delta_{k-1}(\lambda_k) - \delta_{k-1}(z)).$$

Since  $\mathbf{A}(z) = c_1 \Phi(z)$ , it follows that

$$G(z) = c_2 (1 - z\lambda_{n+1}) H(z) = c_2 H(z), \tag{B.5}$$

for some constant  $c_2$ , and that

$$\Phi(z) \begin{bmatrix} 1 \\ -\delta_0(z) \end{bmatrix} = G(z) \begin{bmatrix} 1 \\ -\delta_{n+1}(z) \end{bmatrix}. \tag{B.6}$$

For a function  $f(z)$ , define the tilde operator as  $\tilde{f}(z) = f(1/z)$ . Then,

$$\tilde{\mathbf{A}}_k(z)' \mathbf{A}_k(z) = \left( \delta_{k-1}(\lambda_k)^2 + 1 \right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and, therefore,

$$\tilde{\Phi}(z)' \Phi(z) = c_3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{where} \quad c_3 = \prod_{k=1}^{n+1} \left( \delta_{k-1}(\lambda_k)^2 + 1 \right).$$

Next, apply the tilde transformation to equation (B.6) to obtain

$$\tilde{\Phi}(z) \begin{bmatrix} 1 \\ -\tilde{\delta}_0(z) \end{bmatrix} = \tilde{G}(z) \begin{bmatrix} 1 \\ -\tilde{\delta}_{n+1}(z) \end{bmatrix}.$$

Transposing and multiplying from the right with  $\Phi(z)$  yields

$$c_3 \begin{bmatrix} 1 & -\tilde{\delta}_0(z) \end{bmatrix} = \begin{bmatrix} 1 & -\tilde{\delta}_0(z) \end{bmatrix} \tilde{\Phi}(z)' \Phi(z) = \tilde{G}(z) \begin{bmatrix} 1 & -\tilde{\delta}_{n+1}(z) \end{bmatrix} \Phi(z).$$

Together with the equation (B.6) we obtain four linear equations for the four entries of  $\Phi(z)$ ,

$$\begin{aligned} \Phi_{11}(z) - \Phi_{12}(z)\delta_0(z) &= G(z) \\ \Phi_{21}(z) - \Phi_{22}(z)\delta_0(z) &= -\delta_{n+1}(z)G(z) \\ (\Phi_{11}(z) - \Phi_{21}(z)\tilde{\delta}_{n+1}(z))\tilde{G}(z) &= c_3 \\ (\Phi_{12}(z) - \Phi_{22}(z)\tilde{\delta}_{n+1}(z))\tilde{G}(z) &= -c_3\tilde{\delta}_0(z). \end{aligned}$$

The rank of the system is 3. Use the first, second and fourth equations to express  $\Phi_{11}(z)$ ,  $\Phi_{12}(z)$ ,  $\Phi_{21}(z)$  in terms of  $\Phi_{22}(z)$ . The third equation does not allow to solve for  $\Phi_{22}(z)$ , rather it collapses to

$$\left(1 + \delta_{n+1}(z)\tilde{\delta}_{n+1}(z)\right) G(z)\tilde{G}(z) = \left(1 + \delta_0(z)\tilde{\delta}_0(z)\right) c_3. \quad (\text{B.7})$$

We can, now, determine  $\Phi_{22}(z)$  from<sup>27</sup>

$$\Phi_{11}(z)\Phi_{22}(z) - \Phi_{12}(z)\Phi_{21}(z) = \det(\Phi(z)) = c_3H(z),$$

which implies

$$\Phi_{22}(z) = \frac{(G(z)\tilde{\delta}_0(z)\delta_{n+1}(z) + \tilde{G}(z)H(z))c_3}{(1 + \delta_{n+1}(z)\tilde{\delta}_{n+1}(z))G(z)\tilde{G}(z)}.$$

Together with (B.7), this can be simplified to

$$\Phi_{22}(z) = \frac{G(z)\tilde{\delta}_0(z)\delta_{n+1}(z) + \tilde{G}(z)H(z)}{\delta_0(z)\tilde{\delta}_0(z) + 1}.$$

Applying the tilde operation to equation (B.5) yields

$$\tilde{G}(z) = c_2\tilde{H}(z).$$

Finally, it follows from the definition of  $H(z)$  that  $\tilde{H}(z)H(z) = 1$ , and therefore

$$\Phi_{22}(z) = \frac{c_2H(z)\tilde{\delta}_0(z)\delta_{n+1}(z) + c_2}{\delta_0(z)\tilde{\delta}_0(z) + 1}. \quad \square$$

## C. PROOF OF THEOREM 1

Suppose that when  $\alpha = 0$  the equilibrium is invertible, otherwise the result is trivial. Section C.1 characterizes the equilibrium assuming invertibility. Using this characterization, Section C.2 takes the limit as  $\alpha$  increases to 1 and shows that in it the equilibrium cannot be invertible.

### C.1 Solution Assuming Invertibility

Suppose that the equilibrium is invertible, then the information set of agent  $i$  in period  $t$  is given by  $\mathcal{I}_{it} \equiv \{\mathbf{x}_{i\tau}, \mathbf{z}_\tau, \boldsymbol{\eta}_{\tau-1}, \mathbf{u}_{i\tau-1}\}_{\tau=-\infty}^t$ . Therefore,

$$\mathbb{E}_{it}[\xi_{it}] = \mathbb{E}[\mathbf{d}(L)\boldsymbol{\eta}_t + \mathbf{e}(L)\mathbf{u}_{it}|\mathcal{I}_{it}] = (\mathbf{d}(L) - \mathbf{d}_0)\boldsymbol{\eta}_t + \mathbf{d}_0\mathbb{E}[\boldsymbol{\eta}_t|\mathcal{I}_{it}] + (\mathbf{e}(L) - \mathbf{e}_0)\mathbf{u}_{it} + \mathbf{e}_0\mathbb{E}[\mathbf{u}_{it}|\mathcal{I}_{it}],$$

$$\mathbb{E}_{it}[a_t] = \mathbb{E}[\mathbf{g}(L)\boldsymbol{\eta}_t|\mathcal{I}_{it}] = (\mathbf{g}(L) - \mathbf{g}_0)\boldsymbol{\eta}_t + \mathbf{g}_0\mathbb{E}[\boldsymbol{\eta}_t|\mathcal{I}_{it}].$$

<sup>27</sup>Notice that

$$\det(\mathbf{A}_k) = \left(1 + \delta_{k-1}(\lambda_k)^2\right) \frac{z - \lambda_k}{1 - z\lambda_k}.$$



Moreover, since

$$\mathbf{x}_{it} - (\mathbf{A}(L) - \mathbf{A}_0) \boldsymbol{\eta}_t - (\mathbf{B}(L) - \mathbf{B}_0) \mathbf{u}_{it} = \mathbf{A}_0 \boldsymbol{\eta}_t + \mathbf{B}_0 \mathbf{u}_{it}, \quad \text{and} \quad \mathbf{z}_t - (\mathbf{C}(L) - \mathbf{C}_0) \boldsymbol{\eta}_t = \mathbf{C}_0 \boldsymbol{\eta}_t,$$

it follows that  $\mathbf{A}_0 \boldsymbol{\eta}_t + \mathbf{B}_0 \mathbf{u}_{it}$  is a noisy signal about  $\boldsymbol{\eta}_t$  and  $\mathbf{u}_{it}$ , and  $\mathbf{C}_0 \boldsymbol{\eta}_t$  is a noisy signal about  $\boldsymbol{\eta}_t$ , which allows us to calculate

$$\begin{aligned} \mathbb{E}[\boldsymbol{\eta}_t | \mathcal{I}_{it}] &= \begin{bmatrix} \boldsymbol{\Sigma}_\eta^2 \mathbf{A}_0^\top & \boldsymbol{\Sigma}_\eta^2 \mathbf{C}_0^\top \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{A}_0 \boldsymbol{\Sigma}_\eta^2 \mathbf{A}_0^\top + \mathbf{B}_0 \boldsymbol{\Sigma}_u^2 \mathbf{B}_0^\top & \mathbf{A}_0 \boldsymbol{\Sigma}_\eta^2 \mathbf{C}_0^\top \\ \mathbf{C}_0 \boldsymbol{\Sigma}_\eta^2 \mathbf{A}_0^\top & \mathbf{C}_0 \boldsymbol{\Sigma}_\eta^2 \mathbf{C}_0^\top \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{A}_0 \boldsymbol{\eta}_t + \mathbf{B}_0 \mathbf{u}_{it} \\ \mathbf{C}_0 \boldsymbol{\eta}_t \end{bmatrix}, \\ \mathbb{E}[\mathbf{u}_{it} | \mathcal{I}_{it}] &= \begin{bmatrix} \boldsymbol{\Sigma}_u^2 \mathbf{B}_0^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{A}_0 \boldsymbol{\Sigma}_\eta^2 \mathbf{A}_0^\top + \mathbf{B}_0 \boldsymbol{\Sigma}_u^2 \mathbf{B}_0^\top & \mathbf{A}_0 \boldsymbol{\Sigma}_\eta^2 \mathbf{C}_0^\top \\ \mathbf{C}_0 \boldsymbol{\Sigma}_\eta^2 \mathbf{A}_0^\top & \mathbf{C}_0 \boldsymbol{\Sigma}_\eta^2 \mathbf{C}_0^\top \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{A}_0 \boldsymbol{\eta}_t + \mathbf{B}_0 \mathbf{u}_{it} \\ \mathbf{C}_0 \boldsymbol{\eta}_t \end{bmatrix}. \end{aligned}$$

Therefore,

$$a_{it} = \varphi ((d(L) - d_0) \boldsymbol{\eta}_t + d_0 \mathbb{E}[\boldsymbol{\eta}_t | \mathcal{I}_{it}]) + (e(L) - e_0) \mathbf{u}_{it} + e_0 \mathbb{E}[\mathbf{u}_{it} | \mathcal{I}_{it}] + \alpha ((g(L) - g_0) \boldsymbol{\eta}_t + g_0 \mathbb{E}[\boldsymbol{\eta}_t | \mathcal{I}_{it}]),$$

which can be reorganized as

$$a_{it} = [\varphi (d(L) - d_0) + \alpha (g(L) - g_0)] \boldsymbol{\eta}_t + \varphi (e(L) - e_0) \mathbf{u}_{it} + (\varphi d_0 + \alpha g_0) \mathbb{E}[\boldsymbol{\eta}_t | \mathcal{I}_{it}] + \varphi e_0 \mathbb{E}[\mathbf{u}_{it} | \mathcal{I}_{it}].$$

Consistency requires, in particular, that

$$g(L) = \varphi (d(L) - d_0) + \alpha (g(L) - g_0) + (\varphi d_0 + \alpha g_0) \mathbf{k}_1 \boldsymbol{\Omega} + \varphi e_0 \mathbf{k}_2 \boldsymbol{\Omega},$$

where

$$\mathbf{k}_1 \equiv \begin{bmatrix} \boldsymbol{\Sigma}_\eta^2 \mathbf{A}_0^\top & \boldsymbol{\Sigma}_\eta^2 \mathbf{C}_0^\top \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{k}_2 \equiv \begin{bmatrix} \boldsymbol{\Sigma}_u^2 \mathbf{B}_0^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \text{and} \quad \boldsymbol{\Omega} \equiv \begin{bmatrix} \mathbf{A}_0 \boldsymbol{\Sigma}_\eta^2 \mathbf{A}_0^\top + \mathbf{B}_0 \boldsymbol{\Sigma}_u^2 \mathbf{B}_0^\top & \mathbf{A}_0 \boldsymbol{\Sigma}_\eta^2 \mathbf{C}_0^\top \\ \mathbf{C}_0 \boldsymbol{\Sigma}_\eta^2 \mathbf{A}_0^\top & \mathbf{C}_0 \boldsymbol{\Sigma}_\eta^2 \mathbf{C}_0^\top \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{A}_0 \\ \mathbf{C}_0 \end{bmatrix}.$$

We can rewrite the equation above as

$$(1 - \alpha) g(L) = \varphi d(L) + (\varphi d_0 + \alpha g_0) (\mathbf{k}_1 \boldsymbol{\Omega} - \mathbf{I}) + \varphi e_0 \mathbf{k}_2 \boldsymbol{\Omega}$$

It is useful to replace the lag operator in this equation with an arbitrary complex number  $z$ . Evaluating this equation at  $z = 0$ , for instance, implies the following equilibrium condition,

$$g_0 = \varphi (d_0 \mathbf{k}_1 + e_0 \mathbf{k}_2) \boldsymbol{\Omega} (\mathbf{I} - \alpha \mathbf{k}_1 \boldsymbol{\Omega})^{-1}.$$

Notice that, using the block-matrix inversion formula,  $g_0$  can be rewritten as

$$g_0 = \frac{\varphi}{1 - \alpha} d_0 \mathbf{E} + \varphi (d_0 (\mathbf{I} - \mathbf{E}) \mathbf{D} + e_0 \mathbf{F}) \sum_{j=0}^{\infty} \alpha^j [(\mathbf{I} - \mathbf{E}) \mathbf{D} (\mathbf{I} - \mathbf{E})]^j, \quad (\text{C.1})$$

where

$$\begin{aligned}\mathbf{D} &\equiv \Sigma_\eta^2 \mathbf{A}_0^\top \left( \mathbf{A}_0 (\mathbf{I} - \mathbf{E}) \Sigma_\eta^2 \mathbf{A}_0^\top + \mathbf{B}_0 \Sigma_u^2 \mathbf{B}_0^\top \right)^{-1} \mathbf{A}_0, \\ \mathbf{E} &\equiv \Sigma_\eta^2 \mathbf{C}_0^\top \left( \mathbf{C}_0 \Sigma_\eta^2 \mathbf{C}_0^\top \right)^{-1} \mathbf{C}_0, \\ \mathbf{F} &\equiv \Sigma_u^2 \mathbf{B}_0^\top \left( \mathbf{A}_0 (\mathbf{I} - \mathbf{E}) \Sigma_\eta^2 \mathbf{A}_0^\top + \mathbf{B}_0 \Sigma_u^2 \mathbf{B}_0^\top \right)^{-1} \mathbf{A}_0,\end{aligned}$$

and we have used the fact that  $\mathbf{E}$  is idempotent. Finally, substituting the expression for  $\mathbf{g}_0$  into the equation for  $\mathbf{g}(z)$  we obtain

$$\mathbf{g}(z) = \mathbf{g}_0 + \frac{\varphi}{1 - \alpha} (d(z) - d_0).$$

## C.2 Taking the Limit as $\alpha$ Increases to 1

Let

$$\tilde{\mathbf{g}}(z) = \frac{1 - \alpha}{\varphi} \mathbf{g}_0 + d(z) - d_0,$$

so that

$$\mathbf{g}(z) = \frac{\varphi}{1 - \alpha} \tilde{\mathbf{g}}(z),$$

and notice that this is well defined for all  $\alpha < 1$  and that, if  $\lim_{\alpha \rightarrow 1^-} \det \left[ \mathbf{C}(z) \quad \tilde{\mathbf{g}}(z) \right]^\top$  has an inside root, then there exists  $\alpha < 1$  high enough such that  $\det \left[ \mathbf{C}(z) \quad \mathbf{g}(z) \right]^\top$  is well defined and has an inside root. It is, in fact, sufficient to show that

$$\lim_{\alpha \rightarrow 1^-} \det \left[ \mathbf{C}_0 \quad \tilde{\mathbf{g}}_0 \right]^\top = 0.$$

Accordingly, using equation (C.1) we have that

$$\lim_{\alpha \rightarrow 1^-} \tilde{\mathbf{g}}_0 = \lim_{\alpha \rightarrow 1^-} \frac{1 - \alpha}{\varphi} \mathbf{g}_0 = d_0 \mathbf{E} + \lim_{\alpha \rightarrow 1^-} (1 - \alpha) (d_0 (\mathbf{I} - \mathbf{E}) \mathbf{D} + e_0 \mathbf{F}) \sum_{j=0}^{\infty} \alpha^j [(\mathbf{I} - \mathbf{E}) \mathbf{D} (\mathbf{I} - \mathbf{E})]^j.$$

In order to proceed, it is useful to establish the following lemma.

**Lemma 6.** *The matrix  $(\mathbf{I} - \mathbf{E}) \mathbf{D} (\mathbf{I} - \mathbf{E})$  has all eigenvalues in  $[0, 1)$ .*

*Proof.* Notice that  $(\mathbf{I} - \mathbf{E}) \Sigma_\eta^2$  is symmetric, so that  $\left( (\mathbf{I} - \mathbf{E}) \Sigma_\eta^2 \right)^\top = (\mathbf{I} - \mathbf{E}) \Sigma_\eta^2$ . Let  $\mathbf{M} = \mathbf{A}_0 (\mathbf{I} - \mathbf{E}) \Sigma_\eta$  and  $\mathbf{N} = \mathbf{B}_0 \Sigma_u$ , then it follows that

$$\begin{aligned}\mathbf{M} \mathbf{M}^\top &= \mathbf{A}_0 (\mathbf{I} - \mathbf{E}) \Sigma_\eta \left( (\mathbf{I} - \mathbf{E}) \Sigma_\eta \right)^\top \mathbf{A}_0^\top \\ &= \mathbf{A}_0 (\mathbf{I} - \mathbf{E}) \Sigma_\eta \left( (\mathbf{I} - \mathbf{E}) \Sigma_\eta^2 \Sigma_\eta^{-1} \right)^\top \mathbf{A}_0^\top \\ &= \mathbf{A}_0 (\mathbf{I} - \mathbf{E}) \Sigma_\eta \Sigma_\eta^{-1} \left( (\mathbf{I} - \mathbf{E}) \Sigma_\eta^2 \right)^\top \mathbf{A}_0^\top \\ &= \mathbf{A}_0 (\mathbf{I} - \mathbf{E}) \Sigma_\eta^2 \mathbf{A}_0^\top.\end{aligned}$$

Therefore

$$(\mathbf{I} - \mathbf{E}) \mathbf{D} (\mathbf{I} - \mathbf{E}) = \Sigma_\eta \mathbf{M}^\top (\mathbf{M} \mathbf{M}^\top + \mathbf{N} \mathbf{N}^\top)^{-1} \mathbf{M} \Sigma_\eta^{-1},$$

so that  $(\mathbf{I} - \mathbf{E}) \mathbf{D} (\mathbf{I} - \mathbf{E})$  is similar to  $\mathbf{M}^\top (\mathbf{M} \mathbf{M}^\top + \mathbf{N} \mathbf{N}^\top)^{-1} \mathbf{M}$ . Let  $n \equiv \dim(\mathbf{u}_{it})$  and  $m \equiv \dim(\boldsymbol{\eta}_t)$ , then, it follows that  $\mathbf{N}$  is  $n \times n$  and  $\mathbf{M}$  is  $m - 1 \times m$ . If  $n \geq m$ ,

$$\text{spectrum} \left( \mathbf{M}^\top (\mathbf{M} \mathbf{M}^\top + \mathbf{N} \mathbf{N}^\top)^{-1} \mathbf{M} \right) \cup \{0_1, \dots, 0_{n-m}\} = \text{spectrum} \left( \mathbf{M} \mathbf{M}^\top (\mathbf{M} \mathbf{M}^\top + \mathbf{N} \mathbf{N}^\top)^{-1} \right)$$

while if  $n \leq m$ ,

$$\text{spectrum} \left( \mathbf{M}^\top (\mathbf{M} \mathbf{M}^\top + \mathbf{N} \mathbf{N}^\top)^{-1} \mathbf{M} \right) = \text{spectrum} \left( \mathbf{M} \mathbf{M}^\top (\mathbf{M} \mathbf{M}^\top + \mathbf{N} \mathbf{N}^\top)^{-1} \right) \cup \{0_1, \dots, 0_{m-n}\}.$$

The matrix  $\mathbf{M} \mathbf{M}^\top (\mathbf{M} \mathbf{M}^\top + \mathbf{N} \mathbf{N}^\top)^{-1}$  is the product of a positive semi-definite with a positive definite matrix, so must have positive eigenvalues while  $\mathbf{N} \mathbf{N}^\top (\mathbf{M} \mathbf{M}^\top + \mathbf{N} \mathbf{N}^\top)^{-1}$  is the product of two positive definite matrices, and, therefore, has strictly positive eigenvalues. Finally, since

$$\mathbf{M} \mathbf{M}^\top (\mathbf{M} \mathbf{M}^\top + \mathbf{N} \mathbf{N}^\top)^{-1} = \mathbf{I} - \mathbf{N} \mathbf{N}^\top (\mathbf{M} \mathbf{M}^\top + \mathbf{N} \mathbf{N}^\top)^{-1},$$

it follows that  $\mathbf{M} \mathbf{M}^\top (\mathbf{M} \mathbf{M}^\top + \mathbf{N} \mathbf{N}^\top)^{-1}$  must have eigenvalues lower than 1. Hence,

$$\begin{aligned} \text{spectrum} ((\mathbf{I} - \mathbf{E}) \mathbf{D} (\mathbf{I} - \mathbf{E})) &= \text{spectrum} \left( \mathbf{M}^\top (\mathbf{M} \mathbf{M}^\top + \mathbf{N} \mathbf{N}^\top)^{-1} \mathbf{M} \right) \\ &= \text{spectrum} \left( \mathbf{M} \mathbf{M}^\top (\mathbf{M} \mathbf{M}^\top + \mathbf{N} \mathbf{N}^\top)^{-1} \right) \subset [0, 1). \end{aligned}$$

□

The lemma implies that

$$\sum_{j=0}^{\infty} \alpha^j [(\mathbf{I} - \mathbf{E}) \mathbf{D} (\mathbf{I} - \mathbf{E})]^j$$

is well defined and finite, so that

$$\lim_{\alpha \rightarrow 1^-} (1 - \alpha) (d_0 (\mathbf{I} - \mathbf{E}) \mathbf{D} + e_0 \mathbf{F}) \sum_{j=0}^{\infty} \alpha^j [(\mathbf{I} - \mathbf{E}) \mathbf{D} (\mathbf{I} - \mathbf{E})]^j = \mathbf{0}.$$

Therefore,

$$\lim_{\alpha \rightarrow 1^-} \tilde{\mathbf{g}}_0 = d_0 \mathbf{E},$$

and, using the definition of  $\mathbf{E}$ ,

$$\tilde{\mathbf{g}}_0 = d_0 \boldsymbol{\Sigma}_\eta^2 \mathbf{C}_0^\top \left( \mathbf{C}_0 \boldsymbol{\Sigma}_\eta^2 \mathbf{C}_0^\top \right)^{-1} \mathbf{C}_0 = \mathbf{a} \mathbf{C}_0,$$

for some vector  $\mathbf{a}$ . Finally, notice that

$$\det \begin{bmatrix} \mathbf{C}_0 \\ \tilde{\mathbf{g}}_0 \end{bmatrix} = \det \begin{bmatrix} \mathbf{C}_0 \\ \mathbf{a} \mathbf{C}_0 \end{bmatrix} = 0,$$

which implies that  $z = 0$  is a root of  $\det \begin{bmatrix} \mathbf{C}(z) & \tilde{\mathbf{g}}(z) \end{bmatrix}^\top$  and, therefore,  $\begin{bmatrix} \mathbf{C}(L) & \mathbf{g}(L) \end{bmatrix}^\top$  is not invertible for  $\alpha$  close enough to zero (from below).

## D. PROOF OF PROPOSITION 5.2

The following lemma establishes a type of law of iterated expectations.

**Lemma 7.** *If  $\mathcal{I}_{it} \supseteq \{\eta_\tau, \mathbf{u}_{i\tau}\}_{\tau=-\infty}^{t-1}$ , then, for any stochastic variable  $y_{i,t+j} = \mathbf{f}(L)\boldsymbol{\eta}_{t+j} + \mathbf{g}(L)\mathbf{u}_{i,t+j} = \sum_{s=0}^{\infty} \mathbf{f}_s \boldsymbol{\eta}_{t+j-s} + \sum_{s=0}^{\infty} \mathbf{g}_s \mathbf{u}_{i,t+j-s}$ ,*

$$\mathbb{E}_{it}[\bar{\mathbb{E}}_{t+k}[y_{i,t+j}]] = \mathbb{E}_{it}[y_{i,t+j}], \quad \text{for all } k \geq 1.$$

*Proof.* Let  $\mathbf{f}_s \equiv \mathbf{0}$ , and  $\mathbf{g}_s \equiv \mathbf{0}$ , for all  $s < 0$  and note that

$$\mathbb{E}_{i,t+k}[y_{i,t+j}] = \sum_{s=1}^{\infty} (\mathbf{f}_{j-k+s} \boldsymbol{\eta}_{t+k-s} + \mathbf{g}_{j-k+s} \mathbf{u}_{i,t+k-s}) + \mathbf{f}_{j-k} (\mathbf{H}_\eta \boldsymbol{\eta}_{t+k} + \mathbf{H}_u \mathbf{u}_{i,t+k}) + \mathbf{g}_{j-k} (\mathbf{P}_\eta \boldsymbol{\eta}_{t+k} + \mathbf{P}_u \mathbf{u}_{i,t+k}),$$

where we have used the fact that: (1)  $\mathbb{E}_{i,t+k}[\boldsymbol{\eta}_{t+j}] = \mathbb{E}_{i,t+k}[\mathbf{u}_{i,t+j}] = \mathbf{0}$ , for  $j > k$ ; (2)  $\mathbb{E}_{i,t+k}[\boldsymbol{\eta}_{t+j}] = \boldsymbol{\eta}_{t+j}$ , and  $\mathbb{E}_{i,t+k}[\mathbf{u}_{i,t+j}] = \mathbf{u}_{i,t+j}$ , for  $j < k$ ; and (3)  $\mathbb{E}_{i,t+k}[\boldsymbol{\eta}_{t+k}] = \mathbf{H}_\eta \boldsymbol{\eta}_{t+k} + \mathbf{H}_u \mathbf{u}_{i,t+k}$ , and  $\mathbb{E}_{i,t+k}[\mathbf{u}_{i,t+k}] = \mathbf{P}_\eta \boldsymbol{\eta}_{t+k} + \mathbf{P}_u \mathbf{u}_{i,t+k}$ , for some constant matrices  $\mathbf{H}_\eta$ ,  $\mathbf{H}_u$ ,  $\mathbf{P}_\eta$ , and  $\mathbf{P}_u$ .

In aggregate,

$$\bar{\mathbb{E}}_{t+k}[y_{i,t+j}] = \sum_{s=1}^{\infty} \mathbf{f}_{j-k+s} \boldsymbol{\eta}_{t+k-s} + \mathbf{f}_{j-k} \mathbf{H}_\eta \boldsymbol{\eta}_{t+k} + \mathbf{g}_{j-k} \mathbf{P}_\eta \boldsymbol{\eta}_{t+k}.$$

Consider agent  $i$ 's the inference in period  $t$ ,

$$\mathbb{E}_{it}[\bar{\mathbb{E}}_{t+k}[y_{i,t+j}]] = \sum_{s=1}^{\infty} \mathbf{f}_{j-s} \boldsymbol{\eta}_{t-s} + \mathbf{f}_j (\mathbf{H}_\eta \boldsymbol{\eta}_t + \mathbf{H}_u \mathbf{u}_{it}) = \mathbb{E}_{it}[y_{i,t+j}],$$

where the last equality follows from the same three facts listed above (with  $k = 0$ ). □

Next, substituting the best-response (5.7) into itself, using the law of iterated expectations, we get

$$a_{it} = \mathbb{E}_{it}[\varphi \xi_{it} + (\alpha + \gamma(L))a_t + \beta(L)(\varphi \xi_{it} + (\alpha + \gamma(L))a_t + \beta(L)a_{it})],$$

and iterating on this procedure, using the fact that  $\|\beta(L)\| < 1$  in the operator norm, leads to

$$a_{it} = \mathbb{E}_{it}[\varphi \widehat{\xi}_{it} + (\alpha + \kappa(L))a_t], \quad \text{where } \widehat{\xi}_{it} \equiv \frac{1}{1 - \beta(L)} \xi_{it} \quad \text{and} \quad \kappa(L) \equiv \frac{\gamma(L) + \alpha\beta(L)}{1 - \beta(L)}. \quad (\text{D.1})$$

Notice that  $\kappa_0 = 0$ . Aggregating implies

$$a_t = \bar{\mathbb{E}}_t[\varphi \widehat{\xi}_{it} + (\alpha + \kappa(L))a_t].$$

Multiplying both sides by  $\kappa(L)$  and considering the inference of agent  $i$  in period  $t$ , we have that,

$$\mathbb{E}_{it}[\kappa(L)a_t] = \mathbb{E}_{it} \left[ \kappa(L) \bar{\mathbb{E}}_t[\varphi \widehat{\xi}_{it} + (\alpha + \kappa(L))a_t] \right].$$

Since the equilibrium is invertible,  $\mathcal{I}_{it} \supseteq \{\eta_\tau, \mathbf{u}_{i\tau}\}_{\tau=-\infty}^{t-1}$ , so that, using Lemma 7, it follows that,

$$\mathbb{E}_{it}[\kappa(L)a_t] = \mathbb{E}_{it}[\varphi \kappa(L) \widehat{\xi}_{it} + \kappa(L)(\alpha + \kappa(L))a_t],$$

and, iterating on this procedure, using the fact that  $\|\alpha + \kappa(L)\| < 1$ ,<sup>28</sup> we obtain

$$\mathbb{E}_{it}[\kappa(L)a_t] = \mathbb{E}_{it} \left[ \varphi \frac{\kappa(L)}{1 - (\alpha + \kappa(L))} \widehat{\xi}_{it} \right].$$

The result follows from substituting this fact and the definitions of  $\widehat{\xi}_{it}$  and  $\kappa(L)$  into equation (D.1).

## E. INVERTIBILITY IN NEW KEYNESIAN MODEL WITH INCOMPLETE INFORMATION

### E.1 Firms: The New Keynesian Phillips Curve

The optimal reset price solves the following problem:

$$P_{i,t}^* = \arg \max_{P_{i,t}} \sum_{k=0}^{\infty} (\delta\theta)^k \mathbb{E}_{i,t} [Q_{t|t+k} (P_{i,t} Y_{i,t+k|t} - P_{t+k} mc_{t+k} Y_{i,t+k|t})]$$

subject to the demand equation,  $Y_{i,t+k} = \left(\frac{P_{i,t}}{P_{t+k}}\right)^{-\varepsilon} Y_{t+k}$ , where  $Q_{t|t+k}$  is the stochastic discount factor between  $t$  and  $t+k$ ,  $Y_{t+k}$  and  $P_{t+k}$  are, respectively, aggregate income and the aggregate price level in period  $t+k$ ,  $P_{i,t}$  is the firm's price, as set in period  $t$ ,  $Y_{i,t+k|t}$  is the firm's quantity in period  $t+k$ , conditional on not having changed the price since  $t$ , and  $mc_{t+k}$  is the real marginal cost in period  $t+k$ . The firm's discount factor is  $\delta$ , and  $\theta$  is the Calvo parameter (probability of not resetting price).

Taking the first-order condition and log-linearizing around a steady state with no shocks and zero inflation, we get the following, familiar, characterization of the optimal reset price:

$$p_{i,t}^* = (1 - \delta\theta) \sum_{k=0}^{\infty} (\delta\theta)^k \mathbb{E}_{i,t} [mc_{t+k} + p_{t+k}]. \quad (\text{E.1})$$

Suppose that firms observe the aggregate prices up to period  $t-1$ , that is, they observe  $p^{t-1}$ , then we can restate condition (E.1) as

$$p_{i,t}^* - p_{t-1} = (1 - \delta\theta) \sum_{k=0}^{\infty} (\delta\theta)^k \mathbb{E}_{i,t} [mc_{t+k}] + \sum_{k=0}^{\infty} (\delta\theta)^k \mathbb{E}_{i,t} [\pi_{t+k}]. \quad (\text{E.2})$$

Since only a fraction  $1 - \theta$  of the firms adjust their prices each period, the price level in period  $t$  is given by  $p_t = (1 - \theta) \int p_{i,t}^* di + \theta p_{t-1}$ , and inflation is given by

$$\pi_t \equiv p_t - p_{t-1} = (1 - \theta) \int (p_{i,t}^* - p_{t-1}) di.$$

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<sup>28</sup>To see this, notice that

$$\|\alpha + \kappa(L)\| = \left\| \frac{\alpha + \gamma(L)}{1 - \beta(L)} \right\| \leq \|\alpha + \gamma(L)\| \|(1 - \beta(L))^{-1}\| \leq \|\alpha + \gamma(L)\| \|1 - \beta(L)\|^{-1} \leq \frac{\alpha + \|\gamma(L)\|}{1 - \|\beta(L)\|} < 1,$$

where the first inequality follows from Cauchy-Schwarz, the second from the fact that, for any operator  $T$ ,  $\|T\|^{-1} \leq \|T^{-1}\|$ , and the fourth from the assumptions that  $\|\beta(L)\| < 1$ , and  $|\alpha| + \|\gamma(L)\| + \|\beta(L)\| < 1$ .

Define the firm specific inflation rate to be

$$\pi_{i,t} \equiv (1 - \theta) (p_{i,t}^* - p_{t-1}).$$

Then, it follows from equation (E.2) that

$$\pi_{i,t} = (1 - \theta) \mathbb{E}_{i,t} [(1 - \delta\theta) mc_t + \pi_t] + \delta\theta \mathbb{E}_{i,t} \left[ (1 - \theta) \sum_{k=0}^{\infty} (\delta\theta)^k \mathbb{E}_{i,t+1} [(1 - \delta\theta) mc_{t+1+k} + \pi_{t+1+k}] \right],$$

and we obtain the following beauty contest game, which includes equation (6.1),

$$\pi_{i,t} = (1 - \theta) (1 - \delta\theta) \mathbb{E}_{i,t} [mc_t] + (1 - \theta) \mathbb{E}_{i,t} [\pi_t] + \delta\theta \mathbb{E}_{i,t} [\pi_{i,t+1}], \quad \text{with } \pi_t = \int \pi_{i,t} di.$$

## E.2 Households: The Dynamic IS

The consumer's problem is

$$\max_{\{C_{i,t}, B_{i,t+1}\}} \mathbb{E}_{i,t} \left[ \sum_{k=0}^{\infty} \beta^k \frac{C_{i,t+k}^{1-\frac{1}{\zeta}}}{1 - \frac{1}{\zeta}} \right]$$

subject to

$$C_{i,t} + B_{i,t+1} = R_{t-1} B_{i,t} + Y_t,$$

where  $R_t$  and  $W$  denote real interest rates and income. At any state, the life-time budget constraint can be written as

$$\sum_{k=0}^{\infty} \frac{C_{i,t+k}}{\prod_{j=1}^k R_{t+j-1}} = R_{t-1} B_{i,t} + \sum_{k=0}^{\infty} \frac{Y_t}{\prod_{j=1}^k R_{t+j-1}},$$

which can be log-linearized into

$$\sum_{k=0}^{\infty} \beta^k c_{i,t+k} = \sum_{k=0}^{\infty} \beta^k y_{t+k}.$$

Combining this with the log-linearized version of the households' Euler equation,

$$c_{i,t} = \mathbb{E}_{i,t}[c_{i,t+1}] - \varsigma \mathbb{E}_{i,t}[r_t],$$

and using the market clearing condition,  $y_t = c_t$ , we obtain

$$c_{i,t} = -\varsigma \beta \sum_{k=0}^{\infty} \beta^k \mathbb{E}_{i,t}[r_{t+k}] + (1 - \beta) \sum_{k=0}^{\infty} \beta^k \mathbb{E}_{i,t}[c_{t+k}].$$

Finally, notice this is implied by the following beauty-contest game,

$$c_{i,t} = -\varsigma \beta \mathbb{E}_{i,t}[r_t] + (1 - \beta) \mathbb{E}_{i,t}[c_t] + \beta \mathbb{E}_{i,t}[c_{i,t+1}], \quad \text{with } c_t = \int c_{i,t} di.$$

Letting  $\text{mpc} = 1 - \beta$  yields equation (6.2).

### E.3 Invertibility: Proof of Corollary 2

Suppose that  $\xi_t$  follows an AR(1) process

$$\xi_t = \rho \xi_{t-1} + \eta_t + \theta \eta_{t-1}, \quad \text{with } \eta_t \sim N(0, 1),$$

and that, besides past prices, firms only observe a private signal about it with precision  $\tau_u$ ,

$$x_{i,t} = \xi_t + u_{i,t}, \quad \text{with } u_{i,t} \sim N(0, \tau_u^{-1}).$$

Then, it follows from Corollary 3 that for a best response

$$a_{i,t} = \varphi \mathbb{E}_{i,t} [\xi_t] + \alpha \mathbb{E}_{i,t} [a_t] + \gamma \mathbb{E}_{i,t} [a_{t+1}] + \beta \mathbb{E}_{i,t} [a_{i,t+1}]$$

the equilibrium is invertible if and only if

$$C(\alpha) \equiv \left| \frac{1 - \alpha}{\rho \alpha (1 - H)} \right| > 1, \quad \text{where } H \equiv \frac{\tau_u}{1 + \tau_u}.$$

For positive  $\alpha$  and  $\rho$ , the relevant case here, it is easy to see that  $C(\alpha)$  is a decreasing function. Hence, higher degrees of static strategic complementarity make it less likely that the equilibrium is invertible. Since this is controlled by  $1 - \theta$  in the NKPC and  $\text{mpc} \equiv 1 - \beta$  in the Dynamic IS, invertibility is less likely for lower  $\theta$  and higher  $\text{mpc}$ .

## F. PROOF OF PROPOSITION 6.1

Suppose the equilibrium is invertible and guess

$$\pi_t = \alpha_0 \xi_t^s + \alpha_1 \xi_t^d + \alpha_2 \eta_t^s + \alpha_3 \eta_t^d, \quad \text{and} \quad c_t = \beta_0 \xi_t^s + \beta_1 \xi_t^d + \beta_2 \eta_t^s + \beta_3 \eta_t^d.$$

From the best-response functions, it follows that,

$$\begin{aligned} \pi_t = & \{ \kappa (1 + \beta_0) + (1 - \theta) \alpha_0 + \delta \theta \alpha_0 \rho \} \rho \xi_{t-1}^s + \{ \kappa \beta_1 + (1 - \theta) \alpha_1 + \delta \theta \alpha_1 \rho \} \rho \xi_{t-1}^d \\ & + \{ \kappa \beta_2 + (1 - \theta) \alpha_2 + \kappa (1 + \beta_0) + (1 - \theta) \alpha_0 + \delta \theta \alpha_0 \rho \} E_t [\eta_t^s] \\ & + \{ \kappa \beta_3 + (1 - \theta) \alpha_3 + \kappa \beta_1 + (1 - \theta) \alpha_1 + \delta \theta \alpha_1 \rho \} E_t [\eta_t^d], \end{aligned}$$

and

$$\begin{aligned} c_t = & \{ \text{mpc} \beta_0 - \varsigma (1 - \text{mpc}) (\phi \alpha_0 - \rho \alpha_0) - \rho \beta_0 (\text{mpc} - 1) \} \rho \xi_{t-1}^s \\ & + \{ \text{mpc} \beta_1 - \varsigma (1 - \text{mpc}) (\phi \alpha_1 - \rho \alpha_1 + 1) - \rho \beta_1 (\text{mpc} - 1) \} \rho \xi_{t-1}^d \\ & + \{ \text{mpc} \beta_0 - \varsigma (1 - \text{mpc}) (\phi \alpha_0 - \rho \alpha_0) + \text{mpc} \beta_2 - \rho \beta_0 (\text{mpc} - 1) - \varsigma (1 - \text{mpc}) \phi \alpha_2 \} E_t [\eta_t^s] \\ & + \{ \text{mpc} \beta_1 - \varsigma (1 - \text{mpc}) (\phi \alpha_1 - \rho \alpha_1 + 1) + \text{mpc} \beta_3 - \rho \beta_1 (\text{mpc} - 1) - \varsigma (1 - \text{mpc}) \phi \alpha_3 \} E_t [\eta_t^d]. \end{aligned}$$

Let  $\lambda = \frac{\tau}{1+\tau}$ , then

$$E_t [\eta_t^s] = \lambda \eta_t^s, \text{ and } E_t [\eta_t^d] = \lambda \eta_t^d.$$

Matching coefficients, we get a system of equations that can be split into a system for  $\alpha_0, \beta_0, \alpha_1$ , and  $\beta_1$ ,

$$\begin{aligned} \alpha_0 &= (\kappa (1 + \beta_0) + (1 - \theta) \alpha_0 + \delta \theta \alpha_0 \rho), \\ \beta_0 &= (\text{mpc} \beta_0 - \varsigma(1 - \text{mpc}) (\phi \alpha_0 - \rho \alpha_0) - \rho \beta_0 (\text{mpc} - 1)), \\ \alpha_1 &= (\kappa \beta_1 + (1 - \theta) \alpha_1 + \delta \theta \alpha_1 \rho), \\ \beta_1 &= (\text{mpc} \beta_1 - \varsigma(1 - \text{mpc}) (\phi \alpha_1 - \rho \alpha_1 + 1) - \rho \beta_1 (\text{mpc} - 1)), \end{aligned}$$

a system for  $\alpha_2$ , and  $\beta_2$ ,

$$\begin{aligned} (\alpha_0 + \alpha_2) &= (\kappa \beta_2 + (1 - \theta) \alpha_2 + \kappa (1 + \beta_0) + (1 - \theta) \alpha_0 + \delta \theta \alpha_0 \rho) \lambda, \\ (\beta_0 + \beta_2) &= (\text{mpc} \beta_0 - \varsigma(1 - \text{mpc}) (\phi \alpha_0 - \rho \alpha_0) + \text{mpc} \beta_2 - \rho \beta_0 (\text{mpc} - 1) - \varsigma(1 - \text{mpc}) \phi \alpha_2) \lambda, \end{aligned}$$

and a system for  $\alpha_3$ , and  $\beta_3$ ,

$$\begin{aligned} (\alpha_1 + \alpha_3) &= (\kappa \beta_3 + (1 - \theta) \alpha_3 + \kappa \beta_1 + (1 - \theta) \alpha_1 + \delta \theta \alpha_1 \rho) \lambda, \\ (\beta_1 + \beta_3) &= (\text{mpc} \beta_1 - \varsigma(1 - \text{mpc}) (\phi \alpha_1 - \rho \alpha_1 + 1) + \text{mpc} \beta_3 - \rho \beta_1 (\text{mpc} - 1) - \varsigma(1 - \text{mpc}) \phi \alpha_3) \lambda. \end{aligned}$$

Solving the system yields

$$\begin{aligned} \alpha_0 &= \frac{\kappa (1 - \text{mpc}) (1 - \rho)}{(1 - \text{mpc}) (1 - \rho) (1 - \delta \rho) \theta - \kappa \varsigma (1 - \text{mpc}) (\rho - \phi)}, \\ \alpha_1 &= -\frac{\kappa \varsigma (1 - \text{mpc})}{(1 - \text{mpc}) (1 - \rho) (1 - \delta \rho) \theta - \kappa \varsigma (1 - \text{mpc}) (\rho - \phi)}, \\ \beta_0 &= -\frac{\kappa \varsigma (1 - \text{mpc}) (\phi - \rho)}{(1 - \text{mpc}) (1 - \rho) (1 - \delta \rho) \theta - \kappa \varsigma (1 - \text{mpc}) (\rho - \phi)}, \\ \beta_1 &= -\frac{\theta \varsigma (1 - \text{mpc}) (1 - \delta \rho)}{(1 - \text{mpc}) (1 - \rho) (1 - \delta \rho) \theta - \kappa \varsigma (1 - \text{mpc}) (\rho - \phi)}, \end{aligned}$$

and

$$\begin{aligned} \alpha_2 &= \frac{\kappa (1 - \lambda) ((\text{mpc} - 1) (\rho - 1) (\lambda \text{mpc} - 1) + \kappa \varsigma (1 - \text{mpc}) \lambda \phi - \kappa \varsigma (1 - \text{mpc}) \lambda \rho)}{((1 - \lambda \text{mpc}) (1 - \lambda (1 - \theta)) + \kappa \varsigma (1 - \text{mpc}) \phi \lambda^2) ((1 - \text{mpc}) (1 - \rho) (1 - \delta \rho) \theta - \kappa \varsigma (1 - \text{mpc}) (\rho - \phi))}, \\ \beta_2 &= \frac{\kappa \varsigma (1 - \text{mpc}) (1 - \lambda) (\phi (1 + \lambda (\theta - \text{mpc} - \rho + \text{mpc} \rho)) - \rho (1 - \lambda (1 - \theta)))}{((1 - \lambda \text{mpc}) (1 - \lambda (1 - \theta)) + \kappa \varsigma (1 - \text{mpc}) \phi \lambda^2) ((1 - \text{mpc}) (1 - \rho) (1 - \delta \rho) \theta - \kappa \varsigma (1 - \text{mpc}) (\rho - \phi))}, \\ \alpha_3 &= \frac{\kappa \varsigma (1 - \text{mpc}) (1 - \lambda) (1 + \lambda (\theta - \text{mpc} - \theta \delta \rho))}{((1 - \lambda \text{mpc}) (1 - \lambda (1 - \theta)) + \kappa \varsigma (1 - \text{mpc}) \phi \lambda^2) ((1 - \text{mpc}) (1 - \rho) (1 - \delta \rho) \theta - \kappa \varsigma (1 - \text{mpc}) (\rho - \phi))}, \\ \beta_3 &= \frac{\varsigma (1 - \text{mpc}) (1 - \lambda) (\theta (1 - \delta \rho) (1 - \lambda (1 - \theta)) - \kappa \varsigma (1 - \text{mpc}) \phi \lambda)}{((1 - \lambda \text{mpc}) (1 - \lambda (1 - \theta)) + \kappa \varsigma (1 - \text{mpc}) \phi \lambda^2) ((1 - \text{mpc}) (1 - \rho) (1 - \delta \rho) \theta - \kappa \varsigma (1 - \text{mpc}) (\rho - \phi))}. \end{aligned}$$



This solution validates the guess. Next, to guarantee that the equilibrium is indeed invertible, we need the following system to be invertible,

$$\begin{bmatrix} \pi_t \\ c_t \end{bmatrix} = \begin{bmatrix} \frac{\alpha_0}{1-\rho L} + \alpha_2 & \frac{\alpha_1}{1-\rho L} + \alpha_3 \\ \frac{\beta_0}{1-\rho L} + \beta_2 & \frac{\beta_1}{1-\rho L} + \beta_3 \end{bmatrix} \begin{bmatrix} \eta_t^s \\ \eta_t^d \end{bmatrix}.$$

For that, we need both roots of the determinant,

$$\Delta(L) = \left( \frac{\alpha_0}{1-\rho L} + \alpha_2 \right) \left( \frac{\beta_1}{1-\rho L} + \beta_3 \right) - \left( \frac{\alpha_1}{1-\rho L} + \alpha_3 \right) \left( \frac{\beta_0}{1-\rho L} + \beta_2 \right),$$

to be outside the unit circle. The roots are given by

$$r_1 \equiv -\frac{\tau(1+\theta-\text{mpc}) + \tau\sqrt{(1-\theta-\text{mpc})^2 - 4\kappa\zeta(1-\text{mpc})\phi}}{2\rho},$$

$$r_2 \equiv -\frac{\tau(1+\theta-\text{mpc}) - \tau\sqrt{(1-\theta-\text{mpc})^2 - 4\kappa\zeta(1-\text{mpc})\phi}}{2\rho}.$$

If  $(1-\theta-\text{mpc})^2 - 4\kappa\zeta(1-\text{mpc})\phi < 0$ , the roots are complex and their magnitude is above 1 if

$$\phi > \frac{\rho^2 - \theta(1-\text{mpc})\tau^2}{\kappa\zeta(1-\text{mpc})\tau^2}.$$

Otherwise, the roots are real and  $|r_1| > |r_2|$ . So we need to show that  $|r_2| > 1$ , and for that we need to consider two cases: If  $\tau(1+\theta-\text{mpc}) < 2\rho$ , then  $|r_2|$  is always less than 1, otherwise, it is a necessary and sufficient condition that

$$\phi > \frac{\tau\rho(1+\theta-\text{mpc}) - \rho^2 - \theta(1-\text{mpc})\tau^2}{\kappa\zeta(1-\text{mpc})\tau^2}.$$

## G. FORWARD COMPLEMENTARITY EXAMPLE

Consider the following simple version of equation (5.7),

$$a_{it} = \varphi\mathbb{E}_{it}[\xi_{it}] + \alpha\mathbb{E}_{it}[a_t] + \gamma\mathbb{E}_{it}[a_{t+1}] + \beta\mathbb{E}_{it}[a_{it+1}],$$

and suppose that the equilibrium is invertible. Then, Proposition 5.2 implies that this is equivalent to the static best response

$$a_{it} = \varphi\mathbb{E}_{it}[\tilde{\xi}_{it}] + \alpha\mathbb{E}_{it}[a_t], \quad \text{with} \quad \tilde{\xi}_{it} \equiv \frac{1-\alpha}{1-\alpha-(\gamma+\beta)L^{-1}} \xi_{it}.$$

As noted above, we can see that forward aggregate and individual complementarities, controlled by  $\gamma$  and  $\beta$  have interchangeable effects. This property allows us to simplify the analysis since we do not need to distinguish between the two types of forward complementarities focusing only on their sum.

Next, to be more concrete, suppose that  $\xi_{it}$  does not depend on idiosyncratic shocks (so we suppress the

notation  $i$ ) and follows an ARMA(1,1) process,

$$\xi_t = \rho \xi_{t-1} + \eta_t + \theta \eta_{t-1}, \quad \eta_t \sim \mathcal{N}(0, 1),$$

and that, every period, agent  $i$  observes last period's aggregate action,  $a_{t-1}$ , and a private signal,

$$x_{it} = \xi_t + u_{it}, \quad u_{it} \sim \mathcal{N}(0, \tau_u^{-1}).$$

Then, we can establish the following corollary to Proposition 5.2.

**Corollary 3.** *The equilibrium is invertible if and only if*

$$\left| \frac{1 - \alpha + \theta(\gamma + \beta)}{\rho(1 - \alpha + \theta(\gamma + \beta)) - (\rho + \theta)(1 - \alpha H)} \right| > 1, \quad \text{where } H \equiv \frac{\tau_u}{1 + \tau_u}. \quad (\text{G.1})$$

*Proof.* It follows from Proposition 5.2 that, for any  $d(L)$  such that  $\xi_t = d(L)\eta_t$ , the law of motion for the aggregate action satisfies,

$$g(L) = \varphi \left[ \frac{(1 - \alpha)L}{(1 - \alpha)L - (\gamma + \beta)} d(L) \right]_+ + \alpha(g(L) - g_0) + \alpha g_0 H.$$

where  $[\cdot]_+$  denotes the annihilator operator, and  $H \equiv \tau_u/(1 + \tau_u)$ . Let  $\kappa \equiv (\gamma + \beta)/(1 - \alpha)$ , replace the lag operator in this equation with an arbitrary complex number  $z$ , and rearrange to get

$$(1 - \alpha)g(z) = \varphi \frac{zd(z) - \kappa d(\kappa)}{z - \kappa} + \alpha g_0(H - 1).$$

We can obtain  $g_0 = \varphi d(\kappa)(1 - \alpha H)^{-1}$  by evaluating this equation at  $z = 0$ , and, then,

$$g(z) = g_0 + \frac{\varphi}{1 - \alpha} \frac{d(z) - d(\kappa)}{z - \kappa} z.$$

Using the fact that  $d(z) = (1 + \theta z)/(1 - \rho z)$ , we obtain

$$g(z) = \frac{\varphi}{1 - \rho \kappa} \left( \frac{1 + \theta \kappa}{1 - \alpha H} + \frac{1}{1 - \alpha} \frac{(\rho + \theta)z}{1 - \rho z} \right),$$

which has root

$$z^* = \frac{(1 - \alpha)(1 + \theta \kappa)}{\rho(1 - \alpha)(1 + \theta \kappa) - (\rho + \theta)(1 - \alpha H)}.$$

The equilibrium is invertible if and only if this root to be outside the unit circle, that is  $|z^*| > 1$ .  $\square$

It is easy to see from equation (G.1) that the effect of forward complementarities,  $\gamma + \beta$ , on the invertibility of the equilibrium is ambiguous and depends both on the autoregressive and moving-average parameters,  $\rho$  and  $\theta$ . To interpret this condition, it is useful to consider some particular cases. If, for instance, the fundamental follows an AR(1) process, with  $\theta = 0$ , the inequality simplifies to

$$\left| \frac{1 - \alpha}{\alpha \rho(1 - H)} \right| > 1,$$

and forward complementarities actually do not matter for invertibility.

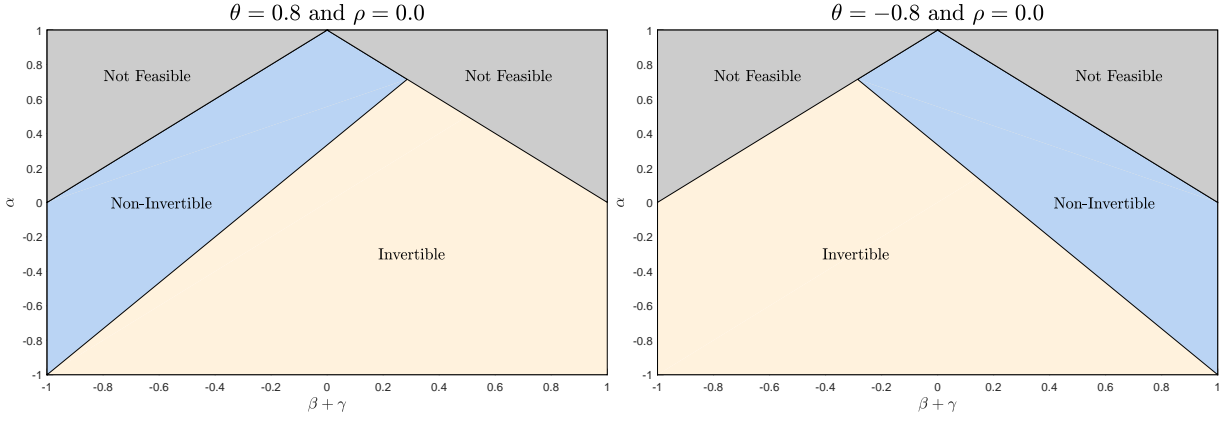


Figure 8: Regions of Invertibility with Forward-Looking Complementarities

The only free parameter,  $\tau_u$ , is set to 1.

To understand this, first let  $[\cdot]_+$  denote the annihilator operator which sets negative powers of the lag operator to zero. Then, because the expected value of future shocks is always zero, we have that, for any stochastic variable  $y_{i,t+j}$  and any  $j$ ,  $\mathbb{E}_{it}[y_{i,t+j}] = \mathbb{E}_{it}[[y_{i,t+j}]_+]$ . When  $\theta = 0$ , we have that

$$[\tilde{\xi}_t]_+ = \left[ \frac{1 - \alpha}{1 - \alpha - (\gamma + \beta)L^{-1}} \frac{1}{1 - \rho L} \eta_t \right]_+ = \frac{1 - \alpha}{1 - \alpha - \rho(\gamma + \beta)} \frac{1}{1 - \rho L} \eta_t.$$

Thus, a change in  $\gamma + \beta$  affects only the variance of the fundamental but not the autoregressive coefficient. Loosely speaking, when  $\beta + \gamma$  increases, the agent puts relatively more weight on the next period fundamental. But since  $\mathbb{E}_{it}[\xi_{t+1}] = \rho \mathbb{E}_{it}[\xi_t]$  this amounts to a proportional increase in the aggregate action in every period which does not affect invertibility.

Things are different if  $\xi_t$  follows an MA(1) process, that is, when  $\rho = 0$  and  $\theta \neq 0$ . In that case, the inequality simplifies to

$$\left| \frac{1 - \alpha + \theta(\gamma + \beta)}{\theta(1 - \alpha H)} \right| > 1,$$

so that, if  $\theta > 0$  ( $< 0$ ) the equilibrium is non-invertible when  $\gamma + \beta$  is low (high) enough.<sup>29</sup> In this case, we have that

$$[\tilde{\xi}_t]_+ = \left[ \frac{1 - \alpha}{1 - \alpha - (\gamma + \beta)L^{-1}} (1 + \theta L) \eta_t \right]_+ = \left( 1 - \frac{\theta(\gamma + \beta)}{1 - \alpha} + \theta L \right) \eta_t.$$

Here, it is useful to consider the response of the aggregate action,  $a_t$ , to a shock to the fundamental,  $\eta_t$ , in period  $t = 0$ , assuming the equilibrium is invertible. Also, for simplicity, suppose that  $\theta > 0$ . Then, an increase in  $\gamma + \beta$  decreases the response of  $a_0$  by an amount proportional to  $\theta$ . On the other hand, it leaves  $a_1$  unchanged since the impulse response of  $\xi_{t+k}$  for  $k \geq 2$  is zero, so that forward-looking complementarities do not affect  $a_1$  or the action in any further period. It follows that  $a_0/a_1$  decreases, which reduces the signal-to-noise ratio and, therefore, the informativeness of the observation of  $a_0$  to forecast  $a_1$ . This, in turn, makes it less likely that the equilibrium is indeed invertible.

Figure 8 shows how when the sign of the moving average parameter,  $\theta$ , flips the effect of an increase in the degree of forward-looking complementarities,  $\beta + \gamma$ , on invertibility. It also illustrates, in accordance with

<sup>29</sup>More specifically, for  $\theta > 0$ , the equilibrium is non-invertible if  $\gamma + \beta < (1 - \alpha H) - (1 - \alpha)/\theta$ , and, for  $\theta < 0$ , if  $\gamma + \beta > -(1 - \alpha H) - (1 - \alpha)/\theta$ .

Theorem 1 and Proposition 5.2, that it is always the case that for a high enough degree of *static* complementarity,  $\alpha$ , the equilibrium is non-invertible.

## H. BACKWARD AND FORWARD COMPLEMENTARITIES

Section 5.2 considers a best response function

$$a_{it} = \varphi \mathbb{E}_{it}[\xi_{it}] + \alpha \mathbb{E}_{it}[a_t] + \delta(L) \mathbb{E}_{it}[a_t] + \lambda(L) \mathbb{E}_{it}[a_{it}], \quad (\text{H.1})$$

with forward looking complementarities, that is, assuming that  $\delta(L)$  and  $\lambda(L)$  are functions only of negative powers of the lag operator  $L$ . This section handles the cases in which backward complementarities as well. First, Section H.1 discusses the case with only static and backward complementarities. Then, Section H.2 deals with the case in which there are both backward and forward complementarities.

$$a_{it} = \varphi \mathbb{E}_{it}[\xi_{it}] + \alpha \mathbb{E}_{it}[a_t] + \delta(L) \mathbb{E}_{it}[a_t] + \lambda(L) \mathbb{E}_{it}[a_{it}],$$

### H.1 Backward Complementarities

Consider the best response in equation H.1 with only backward complementarities, that is, such that  $\delta(L)$  and  $\lambda(L)$  only have positive powers of the lag polynomial. Since past aggregate actions have been assumed to be in agent's information sets, it immediately follows that

$$\mathbb{E}_{it}[a_{t-k}] = a_{t-k}, \quad \text{for all } k \geq 1. \quad (\text{H.2})$$

Assume that the perfect information equilibrium,

$$a_t = \frac{\varphi}{1 - \alpha - \delta(L) - \lambda(L)} \xi_{it}$$

is well defined, that is, that  $\|\alpha + \delta(L) + \lambda(L)\| < 1$  in the operator norm.

**Proposition H.1.** *The equilibrium is invertible with the best response*

$$a_{it} = \varphi \mathbb{E}_{it}[\xi_{it}] + \alpha \mathbb{E}_{it}[a_t],$$

*if and only if it is invertible with the best response in equation H.1 with*

$$\delta(L) = \sum_{k=1}^{\infty} \delta_k L^k, \quad \text{and} \quad \lambda(L) = \sum_{k=1}^{\infty} \lambda_k L^k.$$

*Proof.* It follows from equation (H.2) that the best response with backward complementarities can be rewritten as

$$a_{it} = \varphi \mathbb{E}_{it}[\xi_{it}] + \alpha \mathbb{E}_{it}[a_t] + \delta(L)a_t + \lambda(L)a_{it}.$$

Analogously to the steps in the proof of Theorem 1, we obtain the following consistency requirement for the

law of motion of the aggregate action,

$$\mathbf{g}(L) = \varphi (\mathbf{d}(L) - \mathbf{d}_0) + \alpha (\mathbf{g}(L) - \mathbf{g}_0) + (\varphi \mathbf{d}_0 + \alpha \mathbf{g}_0) \mathbf{k}_1 \mathbf{\Omega} + \varphi \mathbf{e}_0 \mathbf{k}_2 \mathbf{\Omega} + (\delta(L) + \lambda(L)) \mathbf{g}(L),$$

which can be rewritten as

$$(1 - \alpha - \delta(L) - \lambda(L)) \mathbf{g}(L) = \varphi \mathbf{d}(L) + (\varphi \mathbf{d}_0 + \alpha \mathbf{g}_0) (\mathbf{k}_1 \mathbf{\Omega} - \mathbf{I}) + \varphi \mathbf{e}_0 \mathbf{k}_2 \mathbf{\Omega}$$

Since  $\delta_0 = \lambda_0 = 0$ , we have that, just as in the proof of Theorem 1,

$$\mathbf{g}_0 = \varphi (\mathbf{d}_0 \mathbf{k}_1 + \mathbf{e}_0 \mathbf{k}_2) \mathbf{\Omega} (\mathbf{I} - \alpha \mathbf{k}_1 \mathbf{\Omega})^{-1}.$$

and it follows that

$$\mathbf{g}(L) = \frac{(1 - \alpha) \mathbf{g}_0 + \varphi (\mathbf{d}(L) - \mathbf{d}_0)}{1 - \alpha - \delta(L) - \lambda(L)}.$$

Let

$$\tilde{\mathbf{g}}(z) = \frac{1 - \alpha}{\varphi} \mathbf{g}_0 + \mathbf{d}(z) - \mathbf{d}_0,$$

so that

$$\mathbf{g}(z) = \frac{\varphi}{1 - \alpha - \delta(z) - \lambda(z)} \tilde{\mathbf{g}}(z),$$

and notice that  $\tilde{\mathbf{g}}(z)$  is the same as in the proof of Theorem 1, so that, if the equilibrium is invertible in the static best response, it remains invertible with any feasible  $\delta(L)$  and  $\lambda(L)$  since, by assumption,  $\|\alpha + \delta(L) + \lambda(L)\| < 1$ . If it is non-invertible it remains non-invertible for the same reason.  $\square$

It follows that the result in Theorem 1 immediately generalizes to settings with backward-looking complementarities. So that, regardless of these complementarities, if the static complementarity,  $\alpha$ , is large enough the equilibrium is not invertible.

## H.2 Interacting Backward and Forward Complementarities

Next, consider the following best-response function which encompasses most environments considered in the literature including, for instance, the Euler equation in a New-Keynesian model with capital,

$$a_{it} = \varphi \mathbb{E}_{it}[\xi_t] + \alpha \mathbb{E}_{it}[a_t] + \gamma \mathbb{E}_{it}[a_{t+1}] + \beta \mathbb{E}_{it}[a_{it+1}] + \delta \mathbb{E}_{it}[a_{t-1}] + \lambda \mathbb{E}_{it}[a_{it-1}].$$

**Perfect Information Benchmark.** It is easy to see that, if agents observe every shock up to the current period perfectly, the equilibrium must satisfy the following consistency requirement

$$\mathbf{g}(L) = \varphi \mathbf{d}(L) + \alpha \mathbf{g}(L) + (\gamma + \beta) \left( \frac{\mathbf{g}(L) - \mathbf{g}_0}{L} \right) + (\delta + \lambda) \mathbf{g}(L)L,$$

which, replacing the lag operator with an arbitrary complex number  $z$ , can be rewritten as

$$[-(\delta + \lambda)z^2 + (1 - \alpha)z - (\gamma + \beta)] \mathbf{g}(z) = \varphi \mathbf{d}(z)z - (\gamma + \beta) \mathbf{g}_0.$$

In order for this equilibrium to exist and be unique, the polynomial on the left-hand side of this equation must have exactly one inside root, an assumption that we maintain throughout. By inside root we mean that the root is inside the unit circle in the complex plane. The right-hand side of the equation must be zero at any inside root of the polynomial at the left-hand side to avoid poles inside the unit circle. This condition is used to determine  $g_0$ . With two outside roots,  $g_0$  is indeterminate so that there are multiple equilibria, and with two inside roots,  $g_0$  is over-determined so that, in general, an equilibrium does not exist. Let  $\kappa_1$  and  $\kappa_2$  be the inside and outside roots respectively, then, we would have that, the unique perfect-information equilibrium satisfies

$$g(L) = \frac{\varphi}{\delta + \lambda} \frac{\kappa_2^{-1}}{1 - \kappa_2^{-1}L} \frac{d(L)L - d(\kappa_1)\kappa_1}{L - \kappa_1}.$$

In what follows we only consider the set of parameters in which this perfect-information equilibrium exists and is unique, that is, such that  $|\kappa_1| < 1$  and  $|\kappa_2| > 1$ . We refer to parameters that do not satisfy these conditions as infeasible. In this setup, we can establish the following result.

**Theorem 2.** *Suppose that  $e(L) = e$  and  $B(L) = B$ . For any  $\omega_1, \omega_2 \neq 0$  such that  $|\omega_1| < 1$  and  $|\omega_2| < 1$ , there exists  $\varepsilon > 0$  low enough such that if  $\alpha = 1 - (1 + \omega_1\omega_2)\varepsilon$ ,  $\beta + \gamma = \omega_1\varepsilon$ , and  $\delta + \lambda = \omega_2\varepsilon$ , the equilibrium is not invertible.*

*Proof.* The proof is presented in the next Section [H.3](#). □

This theorem extends the result in Theorem [1](#) for the case in which there are both forward and backward complementarities under the restriction that  $e(L) = e$  and  $B(L) = B$ . This restriction is not particularly relevant since most environments considered in the literature do satisfy it. The theorem implies that there always a region in the space of feasible complementarity-parameters  $(\alpha, \gamma, \beta, \delta, \lambda)$  such that the equilibrium is non-invertible. The reason why it is not enough to take the limit as the static degree of complementarity,  $\alpha$ , increases to 1, as in Theorem [1](#), is because, depending on the starting point, that might lead into an infeasible set of parameters. Hence, the limit must be taken in a careful enough way to guarantee that the region of non-invertibility is reached without violating feasibility.

### H.3 Proof of Theorem 2

This proof follows very similar steps to the ones in the proof of Theorem [1](#), for clarity we closely follow the argument of that proof. Suppose that when  $\alpha = 0$  the equilibrium is invertible, otherwise the result is trivial. Section [H.3.1](#) characterizes the equilibrium assuming invertibility. Using this characterization, Section [H.3.2](#) takes the appropriate limit and shows that, in it, the equilibrium cannot be invertible.

#### H.3.1 Solution Assuming Invertibility

Suppose that the equilibrium is invertible, then the information set of agent  $i$  in period  $t$  is given by  $I_{it} \equiv \{\eta_{t-1}, u_{it-1}, x_{it}, z_t\}_{t=-\infty}^t$ . We guess (and verify below) that the individual policy function takes the form

$a_{it} = \mathbf{g}(L)\mathbf{e}a_{it} + \mathbf{h}\mathbf{u}_{it}$ . Therefore,

$$\mathbb{E}_{it} [\xi_{it}] = \mathbb{E} [\mathbf{d}(L)\boldsymbol{\eta}_t + \mathbf{e}\mathbf{u}_{it} | \mathcal{I}_{it}] = (\mathbf{d}(L) - \mathbf{d}_0) \boldsymbol{\eta}_t + \mathbf{d}_0 \mathbb{E} [\boldsymbol{\eta}_t | \mathcal{I}_{it}] + \mathbf{e} \mathbb{E} [\mathbf{u}_{it} | \mathcal{I}_{it}],$$

$$\mathbb{E}_{it} [a_t] = \mathbb{E} [\mathbf{g}(L)\boldsymbol{\eta}_t | \mathcal{I}_{it}] = (\mathbf{g}(L) - \mathbf{g}_0) \boldsymbol{\eta}_t + \mathbf{g}_0 \mathbb{E} [\boldsymbol{\eta}_t | \mathcal{I}_{it}],$$

$$\mathbb{E}_{it} [a_{it+1}] = \mathbb{E} [\mathbf{g}(L)\boldsymbol{\eta}_{t+1} + \mathbf{h}\mathbf{u}_{it+1} | \mathcal{I}_{it}] = \left( \frac{\mathbf{g}(L) - (\mathbf{g}_0 + \mathbf{g}_1 L)}{L} \right) \boldsymbol{\eta}_t + \mathbf{g}_1 \mathbb{E} [\boldsymbol{\eta}_t | \mathcal{I}_{it}],$$

$$\mathbb{E}_{it} [a_{t+1}] = \mathbb{E} [\mathbf{g}(L)\boldsymbol{\eta}_{t+1} | \mathcal{I}_{it}] = \left( \frac{\mathbf{g}(L) - (\mathbf{g}_0 + \mathbf{g}_1 L)}{L} \right) \boldsymbol{\eta}_t + \mathbf{g}_1 \mathbb{E} [\boldsymbol{\eta}_t | \mathcal{I}_{it}],$$

$$\mathbb{E}_{it} [a_{it-1}] = \mathbb{E} [\mathbf{g}(L)\boldsymbol{\eta}_{t-1} + \mathbf{h}\mathbf{u}_{it-1} | \mathcal{I}_{it}] = \mathbf{g}(L)\boldsymbol{\eta}_{t-1} + \mathbf{h}\mathbf{u}_{it-1},$$

$$\mathbb{E}_{it} [a_{t-1}] = \mathbb{E} [\mathbf{g}(L)\boldsymbol{\eta}_{t-1} | \mathcal{I}_{it}] = \mathbf{g}(L)\boldsymbol{\eta}_{t-1}.$$

Moreover, since

$$\mathbf{x}_{it} - (\mathbf{A}(L) - \mathbf{A}_0) \boldsymbol{\eta}_t = \mathbf{A}_0 \boldsymbol{\eta}_t + \mathbf{B}\mathbf{u}_{it}, \quad \text{and} \quad \mathbf{z}_t - (\mathbf{C}(L) - \mathbf{C}_0) \boldsymbol{\eta}_t = \mathbf{C}_0 \boldsymbol{\eta}_t,$$

it follows that  $\mathbf{x}_{it} - (\mathbf{A}(L) - \mathbf{A}_0) \boldsymbol{\eta}_t$  is a noisy signal about  $\boldsymbol{\eta}_t$  and  $\mathbf{u}_{it}$ , and  $\mathbf{z}_t - (\mathbf{C}(L) - \mathbf{C}_0) \boldsymbol{\eta}_t$  is a noisy signal about  $\boldsymbol{\eta}_t$ , which allows us to calculate

$$\mathbb{E} [\boldsymbol{\eta}_t | \mathcal{I}_{it}] = \begin{bmatrix} \boldsymbol{\Sigma}_\eta^2 \mathbf{A}_0^\top & \boldsymbol{\Sigma}_\eta^2 \mathbf{C}_0^\top \end{bmatrix} \begin{bmatrix} \mathbf{A}_0 \boldsymbol{\Sigma}_\eta^2 \mathbf{A}_0^\top + \mathbf{B} \boldsymbol{\Sigma}_u^2 \mathbf{B}^\top & \mathbf{A}_0 \boldsymbol{\Sigma}_\eta^2 \mathbf{C}_0^\top \\ \mathbf{C}_0 \boldsymbol{\Sigma}_\eta^2 \mathbf{A}_0^\top & \mathbf{C}_0 \boldsymbol{\Sigma}_\eta^2 \mathbf{C}_0^\top \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{A}_0 \boldsymbol{\eta}_t + \mathbf{B}\mathbf{u}_{it} \\ \mathbf{C}_0 \boldsymbol{\eta}_t \end{bmatrix},$$

$$\mathbb{E} [\mathbf{u}_{it} | \mathcal{I}_{it}] = \begin{bmatrix} \boldsymbol{\Sigma}_u^2 \mathbf{B}^\top & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{A}_0 \boldsymbol{\Sigma}_\eta^2 \mathbf{A}_0^\top + \mathbf{B} \boldsymbol{\Sigma}_u^2 \mathbf{B}^\top & \mathbf{A}_0 \boldsymbol{\Sigma}_\eta^2 \mathbf{C}_0^\top \\ \mathbf{C}_0 \boldsymbol{\Sigma}_\eta^2 \mathbf{A}_0^\top & \mathbf{C}_0 \boldsymbol{\Sigma}_\eta^2 \mathbf{C}_0^\top \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{A}_0 \boldsymbol{\eta}_t + \mathbf{B}\mathbf{u}_{it} \\ \mathbf{C}_0 \boldsymbol{\eta}_t \end{bmatrix}.$$

Therefore,

$$\begin{aligned} a_{it} &= \varphi ((\mathbf{d}(L) - \mathbf{d}_0) \boldsymbol{\eta}_t + \mathbf{d}_0 \mathbb{E} [\boldsymbol{\eta}_t | \mathcal{I}_{it}] + \mathbf{e} \mathbb{E} [\mathbf{u}_{it} | \mathcal{I}_{it}]) + \alpha ((\mathbf{g}(L) - \mathbf{g}_0) \boldsymbol{\eta}_t + \mathbf{g}_0 \mathbb{E} [\boldsymbol{\eta}_t | \mathcal{I}_{it}]) \\ &\quad + \gamma \left( \left( \frac{\mathbf{g}(L) - (\mathbf{g}_0 + \mathbf{g}_1 L)}{L} \right) \boldsymbol{\eta}_t + \mathbf{g}_1 \mathbb{E} [\boldsymbol{\eta}_t | \mathcal{I}_{it}] \right) + \beta \left( \left( \frac{\mathbf{g}(L) - (\mathbf{g}_0 + \mathbf{g}_1 L)}{L} \right) \boldsymbol{\eta}_t + \mathbf{g}_1 \mathbb{E} [\boldsymbol{\eta}_t | \mathcal{I}_{it}] \right) \\ &\quad + (\delta + \lambda) \mathbf{g}(L) L \boldsymbol{\eta}_t + \lambda \mathbf{h} L \mathbf{u}_{it}, \end{aligned}$$

which can be reorganized as

$$\begin{aligned} a_{it} &= \left[ \varphi (\mathbf{d}(L) - \mathbf{d}_0) + \alpha (\mathbf{g}(L) - \mathbf{g}_0) + (\gamma + \beta) \left( \frac{\mathbf{g}(L) - (\mathbf{g}_0 + \mathbf{g}_1 L)}{L} \right) + (\delta + \lambda) \mathbf{g}(L) L \right] \boldsymbol{\eta}_t \\ &\quad + \lambda \mathbf{h} L \mathbf{u}_{it} + (\varphi \mathbf{d}_0 + \alpha \mathbf{g}_0 + (\gamma + \beta) \mathbf{g}_1) \mathbb{E} [\boldsymbol{\eta}_t | \mathcal{I}_{it}] + \varphi \mathbf{e} \mathbb{E} [\mathbf{u}_{it} | \mathcal{I}_{it}]. \end{aligned}$$

Consistency requires, in particular, that

$$\begin{aligned} \mathbf{g}(L) &= \varphi (\mathbf{d}(L) - \mathbf{d}_0) + \alpha (\mathbf{g}(L) - \mathbf{g}_0) + (\gamma + \beta) \left( \frac{\mathbf{g}(L) - (\mathbf{g}_0 + \mathbf{g}_1 L)}{L} \right) + (\delta + \lambda) \mathbf{g}(L) L \\ &\quad + (\varphi \mathbf{d}_0 + \alpha \mathbf{g}_0 + (\gamma + \beta) \mathbf{g}_1) \mathbf{k}_1 \boldsymbol{\Omega} + \varphi \mathbf{e} \mathbf{k}_2 \boldsymbol{\Omega}, \end{aligned}$$

where

$$\mathbf{k}_1 \equiv \begin{bmatrix} \Sigma_\eta^2 \mathbf{A}_0^\top & \Sigma_\eta^2 \mathbf{C}_0^\top \end{bmatrix}, \quad \mathbf{k}_2 \equiv \begin{bmatrix} \Sigma_u^2 \mathbf{B}_0^\top & \mathbf{0} \end{bmatrix}, \quad \text{and} \quad \Omega \equiv \begin{bmatrix} \mathbf{A}_0 \Sigma_\eta^2 \mathbf{A}_0^\top + \mathbf{B}_0 \Sigma_u^2 \mathbf{B}_0^\top & \mathbf{A}_0 \Sigma_\eta^2 \mathbf{C}_0^\top \\ \mathbf{C}_0 \Sigma_\eta^2 \mathbf{A}_0^\top & \mathbf{C}_0 \Sigma_\eta^2 \mathbf{C}_0^\top \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{A}_0 \\ \mathbf{C}_0 \end{bmatrix}.$$

The guess for the policy function can be verified by collecting the terms associated with the idiosyncratic shocks and noticing that they are zero for any period other than the current one. Thus, we can rewrite the equation above as

$$(1 - \alpha) \left( L - \frac{\gamma + \beta}{1 - \alpha} - \frac{\delta + \lambda}{1 - \alpha} L^2 \right) \mathbf{g}(L) = \varphi \mathbf{d}(L)L - (\gamma + \beta) \mathbf{g}_0 + (\varphi \mathbf{d}_0 + \alpha \mathbf{g}_0 + (\gamma + \beta) \mathbf{g}_1) (\mathbf{k}_1 \Omega - \mathbf{I})L + \varphi \mathbf{e} \mathbf{k}_2 \Omega L \quad (\text{H.3})$$

It is useful to replace the lag operator in this equation with an arbitrary complex number  $z$ . Evaluating this equation at different values of  $z$  implies conditions that allow for the characterization of the equilibrium.

**Solving for  $\mathbf{g}(z)$ .** The right-hand side of equation (H.3) must be equal to  $\mathbf{0}$  when evaluated at the inside root,  $\kappa_1$ , of the second-order polynomial on the left-hand side of the equation; we denote the outside root by  $\kappa_2$ .<sup>30</sup> Moreover, the equation must be consistent with the values of  $\mathbf{g}_0$  and  $\mathbf{g}_1$ . Consistency at  $z = 0$ , i.e.  $\mathbf{g}(0) = \mathbf{g}_0$  is automatic. Next, set  $z = \kappa_1$  to get

$$(1 - \alpha) \mathbf{g}_0 = \varphi \mathbf{d}(\kappa_1) + (\varphi \mathbf{d}_0 + \alpha \mathbf{g}_0 + (\gamma + \beta) \mathbf{g}_1) (\mathbf{k}_1 \Omega - \mathbf{I}) + \varphi \mathbf{e} \mathbf{k}_2 \Omega.$$

It follows that

$$\frac{\mathbf{g}(z) - \mathbf{g}_0}{z} = -\frac{\varphi (\mathbf{d}(z) - \mathbf{d}(\kappa_1))}{(\delta + \gamma)(z - \kappa_1)(z - \kappa_2)} - \frac{\mathbf{g}_0}{z - \kappa_2},$$

and since, by definition,

$$\mathbf{g}_1 = \left. \frac{\mathbf{g}(z) - \mathbf{g}_0}{z} \right|_{z=0},$$

we obtain

$$\mathbf{g}_1 = \frac{\mathbf{g}_0}{\kappa_2} + \frac{\varphi (\mathbf{d}(\kappa_1) - \mathbf{d}_0)}{\gamma + \beta},$$

Putting these results together we obtain that

$$(\kappa_2 - z) \mathbf{g}(z) = \kappa_2 \mathbf{g}_0 + \frac{\varphi}{\gamma + \beta} \frac{\mathbf{d}(z) - \mathbf{d}(\kappa_1)}{z - \kappa_1} z,$$

where

$$\mathbf{g}_0 = \varphi (\mathbf{d}(\kappa_1) \mathbf{k}_1 + \mathbf{e} \mathbf{k}_2) \Omega (\mathbf{I} - (\alpha + (\delta + \lambda) \kappa_1) \mathbf{k}_1 \Omega)^{-1}.$$

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<sup>30</sup>Explicitly,

$$\kappa_1 = \frac{1 - \alpha - \sqrt{(1 - \alpha)^2 - 4(\gamma + \beta)(\delta + \lambda)}}{2(\delta + \lambda)}, \quad \text{and} \quad \kappa_2 = \frac{1 - \alpha + \sqrt{(1 - \alpha)^2 - 4(\gamma + \beta)(\delta + \lambda)}}{2(\delta + \lambda)}.$$



Notice that, using the block-matrix inversion formula,  $\mathbf{g}_0$  can be rewritten as

$$\begin{aligned} \mathbf{g}_0 &= \frac{\varphi}{1 - (\alpha + (\delta + \lambda)\kappa_1)} \mathbf{d}(\kappa_1) \mathbf{E} \\ &\quad + \varphi (\mathbf{d}(\kappa_1) (\mathbf{I} - \mathbf{E}) \mathbf{D} + \mathbf{eF}) \sum_{j=0}^{\infty} (\alpha + (\delta + \lambda)\kappa_1)^j [(\mathbf{I} - \mathbf{E}) \mathbf{D} (\mathbf{I} - \mathbf{E})]^j, \end{aligned} \quad (\text{H.4})$$

where

$$\begin{aligned} \mathbf{D} &\equiv \Sigma_{\eta}^2 \mathbf{A}_0^{\top} \left( \mathbf{A}_0 (\mathbf{I} - \mathbf{E}) \Sigma_{\eta}^2 \mathbf{A}_0^{\top} + \mathbf{B}_0 \Sigma_u^2 \mathbf{B}_0^{\top} \right)^{-1} \mathbf{A}_0, \\ \mathbf{E} &\equiv \Sigma_{\eta}^2 \mathbf{C}_0^{\top} \left( \mathbf{C}_0 \Sigma_{\eta}^2 \mathbf{C}_0^{\top} \right)^{-1} \mathbf{C}_0, \\ \mathbf{F} &\equiv \Sigma_u^2 \mathbf{B}^{\top} \left( \mathbf{A}_0 (\mathbf{I} - \mathbf{E}) \Sigma_{\eta}^2 \mathbf{A}_0^{\top} + \mathbf{B}_0 \Sigma_u^2 \mathbf{B}_0^{\top} \right)^{-1} \mathbf{A}_0, \end{aligned}$$

and we have used the fact that  $\mathbf{E}$  is idempotent.

### H.3.2 Taking the Appropriate Limit

Suppose that  $\alpha = 1 - (1 + \omega_1 \omega_2) \varepsilon$ ,  $\gamma + \beta = \omega_1 \varepsilon$ , and  $\delta + \lambda = \omega_2 \varepsilon$ . It follows that, for all  $\varepsilon > 0$ ,

$$\kappa_1 = \omega_1, \quad \kappa_2 = \omega_2^{-1}, \quad \text{and} \quad \alpha + (\delta + \lambda)\kappa_1 = 1 - \varepsilon.$$

To establish the claim, we consider the limit of

$$\mathbf{g}(z) = \frac{\varphi}{(1 - \omega_2 z) \varepsilon} \left( \frac{\omega_2 \mathbf{d}(z) - \mathbf{d}(\omega_1)}{\omega_1} \frac{z - \omega_1}{z - \omega_1} z + \frac{\varepsilon}{\varphi} \mathbf{g}_0 \right),$$

as  $\varepsilon$  decreases towards 0. Let

$$\tilde{\mathbf{g}}(z) = \frac{\omega_2 \mathbf{d}(z) - \mathbf{d}(\omega_1)}{\omega_1} \frac{z - \omega_1}{z - \omega_1} z + \frac{\varepsilon}{\varphi} \mathbf{g}_0,$$

so that

$$\mathbf{g}(z) = \frac{\varphi}{(1 - \omega_2 z) \varepsilon} \tilde{\mathbf{g}}(z),$$

and notice that this is well defined for all  $\varepsilon > 0$  and that, if  $\lim_{\varepsilon \rightarrow 0^+} \det \left( \left[ \mathbf{C}(z) \quad \tilde{\mathbf{g}}(z) \right]^{\top} \right)$  has an inside root, then there exists  $\varepsilon$  low enough such that  $\det \left( \left[ \mathbf{C}(z) \quad \mathbf{g}(z) \right]^{\top} \right)$  is well defined and has an inside root. Recall that  $|\omega_2| < 1$ . It is, in fact, sufficient to show that

$$\lim_{\varepsilon \rightarrow 0^+} \det \left( \left[ \mathbf{C}(0) \quad \tilde{\mathbf{g}}_0 \right]^{\top} \right) = 0.$$

Accordingly, using equation (H.4) we have that

$$\lim_{\varepsilon \rightarrow 0^+} \tilde{\mathbf{g}}_0 = \lim_{\varepsilon \rightarrow 0^+} \frac{\varepsilon}{\varphi} \mathbf{g}_0 = \mathbf{d}(\omega_1) \mathbf{E} + \lim_{\varepsilon \rightarrow 0^+} \varepsilon (\mathbf{d}(\omega_1) (\mathbf{I} - \mathbf{E}) \mathbf{D} + \mathbf{eF}) \sum_{j=0}^{\infty} (1 - \varepsilon)^j [(\mathbf{I} - \mathbf{E}) \mathbf{D} (\mathbf{I} - \mathbf{E})]^j.$$

It follows from Lemma 6 that

$$\sum_{j=0}^{\infty} (1 - \varepsilon)^j [(\mathbf{I} - \mathbf{E}) \mathbf{D} (\mathbf{I} - \mathbf{E})]^j$$

is well defined and finite, so that

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon (d(\omega_1) (\mathbf{I} - \mathbf{E}) \mathbf{D} + \mathbf{eF}) \sum_{j=0}^{\infty} (1 - \varepsilon)^j [(\mathbf{I} - \mathbf{E}) \mathbf{D} (\mathbf{I} - \mathbf{E})]^j = \mathbf{0}.$$

Therefore,

$$\lim_{\varepsilon \rightarrow 0^+} \tilde{\mathbf{g}}_0 = d(\omega_1) \mathbf{E}$$

and, using the definition of  $\mathbf{E}$ ,

$$\tilde{\mathbf{g}}_0 = d_0 \Sigma_{\eta}^2 \mathbf{C}_0^{\top} \left( \mathbf{C}_0 \Sigma_{\eta}^2 \mathbf{C}_0^{\top} \right)^{-1} \mathbf{C}_0 = \mathbf{a} \mathbf{C}_0$$

for some vector  $\mathbf{a}$ . Finally, notice that

$$\det \begin{bmatrix} \mathbf{C}_0 \\ \tilde{\mathbf{g}}_0 \end{bmatrix} = \det \begin{bmatrix} \mathbf{C}_0 \\ \mathbf{a} \mathbf{C}_0 \end{bmatrix} = 0,$$

which implies that  $z = 0$  is a root of  $\det \begin{bmatrix} \mathbf{C}(z) & \tilde{\mathbf{g}}(z) \end{bmatrix}^{\top}$  and, therefore,  $\begin{bmatrix} \mathbf{C}(L) & \mathbf{g}(L) \end{bmatrix}^{\top}$  is not invertible for  $\varepsilon$  close enough to zero.